# Notes for Algebraic Curves and Riemann Surfaces

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## 1 Complex Analysis

### 1.1 Holomorphic Functions

**Definition 1.1 (Holomorphic function)** A function $f : \mathbb{C} \to \mathbb{C}$ is holomorphic at $z \in \mathbb{C}$ iff the limit

$$\lim_{|h| \to 0} \frac{f(z+h) - f(z)}{h}$$

exists. $f$ is holomorphic on $U \subseteq \mathbb{C}$ if $f$ is holomorphic at every point in $U$.

The condition of holomorphicity is much stronger than real differentiability because $h$ is permitted to approach $z$ through any path. Due to this, a function holomorphic at $z \in \mathbb{C}$ is also analytic (Taylor series of $f$ in a neighborhood of $z$ converges to $f$).

**Theorem 1.2 (Cauchy-Riemann Equations)** *Let $f = u + iv$ be holomorphic on $U \subseteq \mathbb{C}$. Then*

$$u_x = v_y \text{ and } v_x = -u_y$$

*hold on $U$.*

*Proof.*[Proof sketch] Since $f$ is holomorphic, along a vertical and horizontal path we must have the limit equal to $f'(z \in U)$ as $h \to 0$. Vertical path is parameterized by $h = it$ ($t \in \mathbb{R}$) and horizontal path by $h = t$ ($t \in \mathbb{R}$). Equating the limits along the two paths gives the desired result. □

**Definition 1.3 (Orientation & Orientation-preserving)** Let $b_1, b_2$ be bases of a finite-dimensional vector space $V$. Then $b_1, b_2$ have the same orientation if the linear mapping $T : V \to V$ which takes $b_1$ to $b_2$ has positive determinant. This is an equivalence relation on the set of bases of $V$ and has two equivalence classes. An orientation for $V$ is fixing the sign of a class as $+1$ and the other $-1$. Choosing a basis $b$ for $V$ fixes the orientation of $V$ with $[b]$ having sign $+1$. A function $f = u + iv : \mathbb{R}^2 \to \mathbb{R}^2$ is orientation-preserving if on an open dense set of $\mathbb{R}^2$ the Jacobian determinant of $f$, $\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$, is positive.

**Theorem 1.4** *Let $f$ be a non-constant holomorphic function. Viewed as a function $\mathbb{R}^2 \to \mathbb{R}^2$, $f$ is orientation-preserving.*

*Proof.*[Proof sketch] We have $\det J(F) = u_x v_y - v_x u_y$. Cauchy-Riemann equations imply $\det J(F) = u_x^2 + v_x^2$. Since $f$ is non-constant, the result follows. □

**Theorem 1.5 (Rouche's Theorem)** *Suppose $f$ and $g$ are holomorphic in an open set containing a circle $C$ and its interior. If*

$$|f(z)| > |g(z)| \text{ for all } z \in C,$$

*then $f$ and $f + g$ have the same number of zeros inside $C$.*

**Theorem 1.6 (Open Mapping Theorem)** *A non-constant holomorphic function maps open sets to open sets.*

*Proof.* Let $f$ be nonconstant and holomorphic on a region $\Omega \subseteq \mathbb{C}$ and fix $w_0 \in f(\Omega)$ with $f(z_0) = w_0$ for some $z_0 \in \Omega$. Fix a $\delta > 0$ such that the closed disk $\overline{B(z_0, \delta)} \subseteq \Omega$ and for $z \in \partial B(z_0, \delta)$, $f(z) \neq w_0$. Then choose $\epsilon > 0$ such that $|f(z) - w_0)| \geq \epsilon$ for $z \in \partial B(z_0, \delta)$. When $|w - w_0| < \epsilon$, define $g(z) = f(z) - w = (f(z) - w_0) + (w_0 - w) = F(z) + G(z)$; so $|F(z)| > |G(z)|$ on the circle $\partial B(z_0, \delta)$, and thus by Rouche's Theorem $g(z)$ has a zero in $B(z_0, \delta)$ since $F$ has $z_0 \in B(z_0, \delta)$ as a zero. Therefore, $B(w_0, \epsilon) \subseteq f(\Omega)$. □

**Definition 1.7 (Integration along a path)** For an analytic function $f$ and path $\gamma \subset \mathbb{C}$ parameterized by $z : [a, b] \to \mathbb{C}$,

$$\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\gamma'(t)\, dt.$$

**Definition 1.8 (Homotopic paths)** Two paths $\gamma, \nu : [a, b] \to U \subseteq \mathbb{C}$ with coinciding endpoints are said to be related by a continuous deformation of paths or homotopic if there is a continous function $H : [a, b] \times [0, 1] \to U$ such that $H(a, t), H(b, t), (t \in [0, 1])$, are the common endpoints of $\gamma, \nu$, and $H(s, 0) = \gamma(s)$ and $H(s, 1) = \nu(s)$, $(s \in [a, b])$.

**Theorem 1.9 (Continuous deformation preserves intergration)** *Suppose $\gamma, \nu : [a,b] \to U \subseteq \mathbb{C}$ are homotopic, with $U$ is open and connected. Then for any holomorphic function $f$ on $U$, we have*

$$\int_\gamma f(z)\,dz = \int_\nu f(z)\,dz.$$

*Proof.* Let $H(s,t)$ be a continuous deformation of $\gamma$ into $\nu$. Since $H(s,t)$ is continuous on the compact $[a,b]\times[0,1]$, the image of $H$, say $K$, is compact. So, there must exist $\epsilon > 0$ such that every disk of radius $3\epsilon$ centered at any point in $K$ is contained in $U \supseteq K$ (argue by contradiction and use that every sequence in $K$ contains a subsequence that converges to a point in $K$ and that $U^c$ is closed). By the uniform continuity of $H$, choose a $\delta > 0$ such that

$$\sup_{t\in[a,b]} |H(s_1,t) - H(s_2,t)| < \epsilon \text{ whenever } |s_1 - s_2| < \delta.$$

Fix $s_1, s_2$ with $|s_1 - s_2| < \delta$. Choose discs $\{D_0, \ldots, D_n\}$ of radius $2\epsilon$, and consecutive points $\{z_0, \ldots, z_{n+1}\}$ on $H(s_1)$ and $\{w_0, \ldots, w_{n+1}\}$ on $H(s_2)$ such that the discs form a covering for $H(s_1)$ and $H(s_2)$, and $z_i, z_{i+1}, w_i, w_{i+1} \in D_i$. Also, $z_0 = w_0 = H(s_1, 0)$ and $z_{n+1} = w_{n+1} = H(s_1, 1)$. Since a holomorphic function has an anitderivative/primitive on an open disk, we can also let $F_i$ denote the primitive of $f$ on $D_i$. On $D_i \cap D_{i+1}$, $F_i$ and $F_{i+1}$ must differ by a constant, and so

$$F_{i+1}(z_{i+1}) - F_i(z_{i+1}) = F_{i+1}(w_{i+1}) - F_i(w_{i+1}).$$

Thus,

$$\begin{aligned}\int_{H(s_1)} f(z)\,dz - \int_{H(s_2)} f(z)\,dz &= \sum_{i=1}^{n}[F_i(z_{i+1}) - F_i(z_i)] + \sum_{i=1}^{n}[F_i(w_{i+1}) - F_i(w_i)] \\ &= F_n(z_{n+1}) - F_n(w_{n+1}) - (F_0(z_0) - F_0(w_0)).\end{aligned}$$

Since $w_0 = z_0$ and $z_{n+1} = w_{n+1}$, we have

$$\int_{H(s_1)} f(z)\,dz = \int_{H(s_2)} f(z)\,dz.$$

Dividing $[0,1]$ into subintervals of length less than $\delta$, by a finite applications of the above argument we can conclude the desired result. □

**Theorem 1.10 (Existence of primitives)** *Any holomorphic function in a simply connected domain has a primitive.*

*Proof.* Stein Shakarchi pp. 96. □

**Definition 1.11 (Simply connected region)** A simply connected region is an open set in which any two paths with the same endpoints are homotopic.

**Corollary 1.11.1** *If $f$ is holomorphic in a simply connected region $\Omega$, then*

$$\int_\gamma f(z)\, dz = 0,$$

*for any closed path $\gamma \subseteq \Omega$.*

*Proof.* Since $f$ has a primitive in $\Omega$ by Theorem 1.10, and $\gamma$ has is a closed path, the result follows. □

**Theorem 1.12 (Cauchy's Integral Formula)** *Let $\gamma$ be a small loop around $z \in \mathbb{C}$ and $f(z)$ a holomorphic function in a neighborhood $U$ of $\gamma$. Then*

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{w - z}\, dw.$$

**Corollary 1.12.1 (Holomorphic functions are analytic)** *Suppose $f$ is holomorphic in an open set $\Omega$. If $D$ is a disk centered at $z_0$ and whose closure is contained in $\Omega$, then $f$ has a power series expansion at $z_0$*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

*for all $z \in D$, and the coefficients are given by*

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(w)\, dw}{(w - z_0)^{n+1}}.$$

*Proof.* Since $z \in D$ is fixed and $w \in C$, we have

$$\left| \frac{z - z_0}{w - z_0} \right| < 1,$$

and so

$$\frac{1}{1 - \frac{z-z_0}{w-z_0}} = \sum_{n=0}^{\infty} (\frac{z - z_0}{w - z_0})^n,$$

with *uniform* convergence guaranteed for $w \in C$. Thus,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_C \frac{f(w)}{w - z_0} \cdot \frac{1}{1 - \frac{z-z_0}{w-z_0}}\, dw \\ &= \frac{1}{2\pi i} \oint_C \frac{f(w)}{w - z_0} \sum_{n=0}^{\infty} (\frac{z - z_0}{w - z_0})^n\, dw \\ &= \sum_{n=0}^{\infty} \left( \frac{1}{2\pi i} \oint_C \frac{f(w)\, dw}{(w - z_0)^{n+1}} \right) (z - z_0)^n. \end{aligned}$$

□

**Definition 1.13 (Pole)** A complex function $f$ has a pole of order $n \in \mathbb{Z}^+$ at $z_0 \in \mathbb{C}$ if $(z-z_0)^n f(z)$ is holomorphic at $z_0$ but $(z-z_0)^{n-1} f(z)$ is not.

**Definition 1.14 (Residue)** Let $f$ have a pole of order $n$ at the point $z_0 \in \mathbb{C}$. Then the residue of $f$ at $z_0$ is the $k=-1$ coefficient in the Laurent expansion of $f$ at $z_0$.

**Theorem 1.15 (Inverse Function Theorem)** *Let $f : U \to \mathbb{C}$ be a holomorphic function and $z_0 \in U$ such that $f'(z_0 \neq 0)$. Then there exists a neighborhood $V$ of $f(z_0)$ and a holomorphic function $g : V \to \mathbb{C}$ such that $z_0 \in g(V)$ and for every $z \in g(V), g(f(z)) = z$.*

*Proof.* □

## 1.2 k-th Roots

The function $f : \mathbb{C} \to \mathbb{C}$ given by $z \mapsto z^k$ has $k$ elements in the preimage at every point other than $0$ in $\mathbb{C}$. Since $f'(z) = kz^{k-1}$, the Inverse Function theorem implies that there exists a holomorphic function $f_{z_0}^{-1}$ at every $z_0 \neq 0$ such that $f_0^{-1}$ is the local inverse of $f$ near $z_0$. This function $f_{z_0}^{-1}$ is called a **branch** of the $k$-th root function $g(z) = z^{1/k}$. But traversing a small circle around $0$ shows that a continuous inverse cannot be constructed for even $\mathbb{C} \setminus \{0\}$.

Riemann's method to fix this was to consider the graph of the $k$-th power function:

$$\Gamma_k = \{(z, w) \in \mathbb{C}^2 \mid w = z^k\},$$

from which a local branch can be chosen using the first coordinate projection $\pi_1 : \Gamma_k \setminus \{(0,0)\} \to \mathbb{C}$, $(z, w) \mapsto z$. This is called the **Riemann Surface of the $k$-th root**.

# 2 Manifolds

## 2.1 Basic Defintions

**Definition 2.1 (Smooth function)** A function $f = (f_1, \dots, f_n) : \mathbb{R}^m \to \mathbb{R}^n$ is smooth if all partial derivatives $\partial^k f_i / \partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_k}$ exist.

**Definition 2.2 (Smooth Manifold)** A Hausdorff topological space $X$ is called a smooth manifold iff

1. For all $x \in X$ there exists a (open) neighborhood $U_x \subseteq X$ of $x$ and a homeomorphism $\varphi_X : U_x \to V_x$, where $V_X \subseteq \mathbb{R}^n$ is open.

2. For any $U_x, U_y$ such that $U_x \cap U_y \neq \varnothing$, the transition function

$$T_{y,x} : \varphi_y \circ \varphi_x^{-1} : \varphi_x(U_x \cap U_y) \to \varphi_y(U_x \cap U_y)$$

is smooth. In this case $\varphi_x$ and $\varphi_y$ are said to be compatible.

The pair $(U_x, \varphi_x)$ is a called a **local chart** and $\varphi_X$ a **local coordinate function**. A collection $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ of local charts that cover $X$ and having smooth transition functions is called an **atlas** for $X$.

**Definition 2.3 (Compatible atlas)** Two atlases $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_\alpha$ and $\mathcal{B} = \{(U_\beta, \varphi_\beta)\}_\beta$ for a smooth manifold $X$ are said to be compatible if $\mathcal{A} \cup \mathcal{B}$ forms an atlas for $X$: whenever $U_\alpha \cap U_\beta \neq \varnothing$, the transition functions $\varphi_\alpha \circ \varphi_\beta^{-1}$ and $\varphi_\beta \circ \varphi_\alpha^{-1}$ are smooth.

Compatibility is an equivalence relation on the set of atlases on a smooth manifold $X$, and an equivalence class of compatible atlases for $X$ is a **smooth differentiable structure** on $X$.

**Example 2.4** The unit circle $S^1 \subset \mathbb{R}^2$ can be a smooth manifold of dimension 1 by equipping it with the topology induced by $\mathbb{R}^2$ (making it Hausdorff) and by letting the coordinate domains be

$U_{x^+} = \{(x, y) \in S^1 \mid x > 0\}, U_{x^-} = \{(x, y) \in S^1 \mid x < 0\}, U_{y^+} = \{(x, y) \in S^1 \mid y > 0\}$, and $U_{y^-} = \{(x, y)$

The coordinate functions can be defined as the projections:

$$\varphi_{x^\pm} : U_{x^\pm} \to (-1, 1)[(x, y) \mapsto y] \text{ and } \varphi_{y^\pm} : U_{y^\pm} \to (-1, 1)[(x, y) \mapsto x].$$

## 2.2 Projective Spaces

**Definition 2.5 (Points of $\mathbb{P}^n(\mathbb{R})$)** The set of points of $\mathbb{P}^n(\mathbb{R})$ are the set of equivalence classes under the relation $\sim$ on $\mathbb{R}^{n+1} \setminus \{\vec{0}\}$ given by $x \sim y$ iff $x = \lambda y$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. Equivalently, they are the set of lines $l$ in $\mathbb{R}^{n+1}$ through the origin.

Given $l \in \mathbb{P}^n(\mathbb{R})$ and $(x_0, \ldots, x_n) \in l$, we denote

$$l = [x_0 : x_1 : \cdots : x_n].$$

**Definition 2.6 (Topology of $\mathbb{P}^n(\mathbb{R}^n)$)** Let $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ act on $\mathbb{R}^{n+1}$ via component-wise multiplication. Then, let $\mathbb{P}^n(\mathbb{R})$ have the quotient topology given by

$$\mathbb{P}^n(\mathbb{R}) = (\mathbb{R}^{n+1} \setminus \{\vec{0}\})/\mathbb{R}^*.$$

In other words, letting $\pi : \mathbb{R}^{n+1} \to \mathbb{P}^n(\mathbb{R})$ be the projection $(x_0, \ldots, x_n) \mapsto [x_0 : x_1 : \cdots : x_n]$, $U \subseteq \mathbb{P}^n(\mathbb{R})$ is open iff $\pi^{-1}(U)$ is open in $\mathbb{R}^{n+1} \setminus \{0\}$.

**Definition 2.7 (Manifold structure of $\mathbb{P}^n(\mathbb{R})$)** Denoting points of $\mathbb{P}^n(\mathbb{R})$ by $[X_0 : X_2 : \cdots : X_n]$, the coordinate domains are defined by

$$U_{X_i} = \{[X_0 : \cdots : X_n] \in P^n(\mathbb{R}) \mid X_i \neq 0\},$$

and coordinate functions $\varphi_{X_i} : U_{X_i} \to \mathbb{R}^n$ by

$$\varphi_{X_i}([X_0 : \cdots : X_n]) = (X_0/X_i, \ldots, X_{i-1}/X_i, X_{i+1}/X_i, \ldots, X_n/X_i),$$

for $i = 0, \ldots, n$.

## 2.3 Compact Surfaces

**Definition 2.8 (Surface)** A surface is a manifold of real dimension $2$.

**Definition 2.9 (Connected Sum)** The connected sum $S_1 \# S_2$ of two connected surfaces $S_1$ and $S_2$ is the surface obtained by removing an open disk from each of the surfaces and identifying the resulting boundaries via a homeomorphism.

The proof that the connected sum $S_1 \# S_2$ is well-defined is non-trivial and requires application of the Annulus Theorem.

**Theorem 2.10 (Classification of Compact Surfaces)** *Any connected, compact surface is homeomorphic to exactly one surface in the following list, with $g, m \in \mathbb{Z}^+$:*

1. *Two-sphere $S^2$,*
2. *$T^{\#g} = T \# \ldots \# T$, the connected sum of $g$ tori,*
3. *$\mathbb{P}^2(\mathbb{R})^{\#m} = \mathbb{P}^2(\mathbb{R}) \# \ldots \# \mathbb{P}^2(\mathbb{R})$, the connected sum of $m$ projective planes.*

Again the proof is non-trivial and makes use of the topological invariants orientability and Euler characteristic.

**Definition 2.11 (Identification Polygon)** A set $A$ of $n$ letters is called an alphabet and $A \cup \overline{A} = A \cup \{\overline{x} \mid x \in A\}$ is called a doubled alphabet. A pair $x, \overline{x}$, for $x \in A$, is called a pair of twin letters. An identification polygon of $2n$ sides is a word $w$ constructed using $2n$ letters from a doubled alphabet such that exactly two letters (with repetition allowed) from each twin pair appear. This word is used to label the edges of a regular $2n$-gon.

An identification polygon $w$ can be used to give a compact surface by identifying the two sides labelled by members of a twin pair via a homeomorphism, reversing orientation if the letters are same and preserving orientation otherwise.

**Definition 2.12 (Good graph/Triangulation)** A triangulation on a surface $S$ is a graph $\Gamma$ on $S$ such that

1. $S \setminus \Gamma$ is homeomorphic to a disjoint union of open disks.
2. Wherever two edges cross there is a vertex.
3. No edge ends without a vertex.

**Definition 2.13 (Euler characteristic)** For any good graph $\Gamma$ on a surface $S$, the Euler characteristic of $\Gamma$ is

$$\chi(S) = |V_\Gamma| - |E_\Gamma| + |F_\Gamma|\,,$$

where $V_\Gamma, E_\Gamma, F_\Gamma$ are the vertices, edges, and faces of $\Gamma$.

$\chi(S)$ is independent of the choice of good graph $\Gamma$. For example, $\chi(\mathbb{R}^2) = 2$ (Euler's formula) and $\chi(S^2) = 2$.

**Definition 2.14 (Orientable Surface)** A surface $S$ is orientable if it admits an atlas such that all transition functions are orientation-preserving and such an atlas is called a positive atlas for $S$.

The Möbius strip is non-orientable while the sphere, the torus, and all connected sums of tori are orientable surfaces.

**Theorem 2.15 (Implicit Function Theorem)** *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth function, and $x \in \mathbb{R}^n$ such that the differential $dF(x)$ is a surjective linear function. Say $f(x) = a$. Then there exist*

1. *an open neighborhood $V_x \subseteq \mathbb{R}^n$ of $x$,*
2. *an open set $U_x \subseteq R^{n-m}$,*
3. *$f_x : U_x \to \mathbb{R}^m$ a smooth function*

*such that*

$$\Gamma_{f_x} = F^{-1}(a) \cap V_x,$$

*where $\Gamma_f$ is the graph of $f$.*

**Definition 2.16 (Regular value)** Let $F : \mathbb{R}^n \to \mathbb{R}^m$ br a smooth function. A point $a \in \mathbb{R}^m$ is called a regular value for $F$, if for every $x$ such that $f(x) = a$, the differential $df(x)$ is a surjective linear function.

**Theorem 2.17 (Manifolds as level sets)** *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth function and $a \in \mathbb{R}^m$ a regular value for $F$. Then $F^{-1}(a)$ is a smooth manifold.*

# 3 Introduction to Riemann Surfaces

## 3.1 Definitions

**Definition 3.1 (Riemann Surface)** A Riemann Surface is a second-countable complex analytic connected manifold of dimension 1.

Thus, the transition functions of an atlas on a Riemann Surface are required to be holomorphic and the local coordinate functions map to $\mathbb{C}$. Since holomorphic functions preserve orientation, every atlas is positive and so a Riemann Surface is orientable.

**Definition 3.2 (Sub-chart)** Let $\phi : U \to V$ be a complex chart on a topological space $X$. Suppose $U' \subseteq U$ is an open subset of $U$. Then $\phi\,|_{U'}: U_1 \to \phi(U_1)$ is a complex chart on $X$ and is called a sub-chart of $\phi$.

**Lemma 3.3** *Let $T$ be a transition function between two compatible charts. Then the derivative $T'$ is never zero on the domain of $T$.*

*Proof.* Since $T$ is bijective, we can consider $S = T^{-1}$ on the domain of $T$. So, $S(T(w)) = w$ on the domain of $T$. Because $T$ is holomorphic, we can differentiate this equation to get $S'(T(w))T'(w) = 1$, which implies $T'(w) \neq 0$ on the domain of $T$. $\square$

An immediate consequence of this is that the second coefficient of the Taylor series expansion of $T$ on its domain must be nonzero.

**Definition 3.4 (Complex structure)** A complex structure on $X$ is a maximal complex atlas on $X$, or equivalently, an equivalence class of complex atlases on $X$.

**Example 3.5 (Riemann Sphere)** The two-sphere $S^2 \subseteq \mathbb{R}^3$ along with the two charts of stereographic projection onto the complex plance from the north pole and south pole give an atlas for $S^2$, which makes $S^2$ into a Riemann surface called the Riemann Sphere, which is also compact.

**Definition 3.6 (Genus of Compact Riemann surface)** Since $\mathbb{P}^2(\mathbb{R})^{\#m}, m \geq 1$, is non-orientable, by classification of compact surfaces, every compact Riemann surface is homeomorphic to a $g$-holed torus $T^{\#g}, g \geq 0$, and $g$ is called the genus of the Riemann surface.

The Riemann surface has genus 0.

## 3.2 Examples of Riemann surfaces

**Example 3.7 (Complex Projective Line)** Defined analogously as $\mathbb{P}^1(\mathbb{R})$, $\mathbb{P}^1(\mathbb{C})$ is the set of 1-dimensional linear subspaces of $\mathbb{C}^2$.

**Example 3.8 (Complex Tori)** Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$ and let $L$ be the lattice

$$L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2.$$

Considering $L$ as a subgroup of the additive group $\mathbb{C}$, let $X = \mathbb{C}/L$, and let $\pi : \mathbb{C} \to X$ be the canonical projection. Put the quotient topology $\mathbb{C}/\pi$ on $X$. Fixing $z_0 \in \mathbb{C}$, choose $\epsilon > 0$ such that $2\epsilon < |\omega|$ for all $\omega \in L$. Then, let the coordinate map $\phi_{z_0} : \pi(B(z_0, \epsilon)) \to B(z_0, \epsilon)$ be the inverse of the map $\pi \mid_{B(z_0,\epsilon)}$. This construction ensures $\varphi_{z_0}$ is a homeomorphism. Details of showing charts are compatible is in Miranda pp. 9. The construction is homeomorphic to a simple torus by considering $X$ as the image of $P_{z_0}$ under $\pi \mid_{P_{z_0}}$, where $z_0 \in \mathbb{C}$ is arbitrary and

$$P_{z_0} = \{z_0 + \lambda_1\omega_1 + \lambda_2\omega_2 : \lambda_i \in [0, 1]\}.$$

**Example 3.9 (Graphs of Holomorphic functions)** Let $V \subseteq \mathbb{C}$ be connected and let $g$ be holomorphic on $V$. Then, the graph of $g$

$$X = \{(z, g(z)) : z \in V\},$$

can be made into a Riemann surface simply by letting the global coordinate function be projection in the first coordinate ($(z, w) \mapsto z$). The fact that $g$ is holomorphic is used in proving that the projection is continuous (in order to show it is homeomorphic).

We can extend this construction of treating graphs as Riemann surfaces by looking at "sufficiently nice" functions locally as graphs of holomorphic functions.

**Theorem 3.10 (Complex version of Implicit Function theorem)** *Let $f(z, w) : \mathbb{C}[z, w]$ be a polynomial, and let $X$ be its zero locus. Let $p = (z_0, w_0) \in X$. Suppose $\frac{\partial f}{\partial w}(p) \neq 0$. Then, there exists a function $g(z)$ holomorphic in a neighborhood of $z_0$ such that near $p$, $X$ is equal to the graph of $g(z)$. Further, $g' = -\frac{\partial f}{\partial z} / \frac{\partial f}{\partial w}$.*

**Example 3.11 (Smooth affine plane curves)** An affine plane curve is the zero locus of a polynomial $f(z, w) \in \mathbb{C}[z, w]$. A polynomial $f(z, w)$ is nonsingular at a root $p$ if either $\frac{\partial f}{\partial w}(p)$ or $\frac{\partial f}{\partial z}(p)$ is nonzero. An affine plane curve is called smooth/nonsingular if $f$ is nonsingular at each of the points on the curve.

Using the Implicit Function theorem, we can locally treat a smooth affine plane curve $X$ as a graph of a holomorphic function and repeat the construction of the previous example to get an atlas for $X$. The only problem is ensuring $X$ is connected, which in particular holds if $f(z, w)$ is an irreducible polynomial (nontrivial fact, uses algebraic geometry). Also, if $X$ is not necessary smooth, we can only look at its smooth part, i.e., ignore the singular points, to get a Riemann surface (assuming $f$ is irreducible). Since the zero locus is not bounded, no affine plane curve is compact.

**Example 3.12 (Smooth projective plane curve)** Let $F(x, y, z)$ be a homogeneous polynomial. Then, the zero locus of $F(x, y, z)$ in $\mathbb{P}^2(\mathbb{C})$ is well-defined:

$$X = \{[x : y : z] \in \mathbb{P}^2(\mathbb{C}) : F(x, y, z) = 0\}.$$

Note that with the coordinate patches $\{U_{x_i}\}_{x_i}$ defined like for real projective plane, we have

$$X_0 = X \cap U_x \cong \{(a,b) \in \mathbb{C}^2 : F(1,a,b) = 0\},$$

which is the affine plane curve for $f(a,b) = F(a,b,1)$ (when transported to $\mathbb{C}^2$). So $X$ is called the projective plane curve for $F$ and we can make this into a Riemann surface by imposing some additional non-singularity conditions on $F$. A homogeneous polynomial $F(x,y,z)$ is *nonsingular* if there are no common solutions to the system

$$F = \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$$

in $\mathbb{P}^2(\mathbb{C})$.

For a homogeneous polynomial $F(x_1, \dots, x_n)$ of degree $d$, we also have the Euler formula

$$F = \frac{1}{d} \sum_i x_i \frac{\partial F}{\partial x_i}.$$

**Lemma 3.13** *Suppose $F(x,y,z)$ is a homogeneous polynomial of degree $d$. Then $F$ is nonsingular iff each $X_i$ is a smooth affine plane curve in $\mathbb{C}^2$.*

*Proof.* Suppose $F$ is nonsingular and one of the $X_i's$, say $X_0$, is not smooth. Let $f(z,w) = F(1,z,w)$. Then, there exists a common solution $(z_0, w_0) \in \mathbb{C}^2$ satisfying

$$f = \frac{\partial f}{\partial z} = \frac{\partial f}{\partial w} = 0.$$

So, $F[1 : z_0 : w_0] = f(z_0, w_0) = 0$, and similarly $\frac{\partial F}{\partial y} = 0 = \frac{\partial F}{\partial z}$. By Euler's formula, we have

$$\frac{\partial F}{\partial x} = dF[1 : z_0 : w_0] - z_0 \frac{\partial F}{\partial y}[1 : z_0 : w_0] - w_0 \frac{\partial F}{\partial z}[1 : z_0 : w_0] = 0,$$

contradicting nonsingularity of $F$. Similarly for the converse. □

By algebraic geometry, a homogeneous nonsingular polynomial is irreducible, allowing us to use the construction of the smooth affine plane curves locally on each of the coordinate patches $U_{x_i}$ to make a homogeneous nonsingular projective plane curve into a Riemann surface. Moreover, since $\mathbb{P}^2$ is compact and the zero locus is a closed subset, we get a compact Riemann surface.

**Theorem 3.14** *Let $F(x,y,z)$ be a nonsingular homogeneous polynomial. Then the projective plane curve $X$, which is the zero locus of $F$ in $\mathbb{P}^2(\mathbb{C})$, is a compact Riemann surface.*

## 3.3 Functions and Maps on Riemann Surfaces

Let $X$ be a Riemann surface, let $p \in X$, and let $f$ be a complex valued function.

**Definition 3.15 (Holomorphic function)** $f$ is holomorphic if there exists a chart $(\phi, U)$, with $p \in U$, such that $f \circ \phi^{-1}$ is holomorphic at $\phi(p)$.

In fact, if $f$ is holomorphic w.r.t. to one chart $(\phi, U)$ at $p$, then $f$ is also holomorphic w.r.t any other chart $(\phi', U')$ containing $p$:

$$f \circ \phi'^{-1} = (f \circ \phi^{-1}) \circ (\phi \circ \phi'^{-1}),$$

which is holomorphic being the composition of holomorphic maps.

**Example 3.16** Consider $\mathbb{P}^1(\mathbb{C})$. Then if $g, h$ are homgeneous polynomials with the same degree and $h(z_0, w_0) \neq 0$, then $F([z, w]) = g(z, w)/h(z, w)$ is a holomorphic function in a neighborhood of $[z_0, w_0]$.

**Example 3.17** Consider a projective plane curve $X$. Then, if $p = [x_0, y_0, z_0]$ with $x_0 \neq 0$, then any polynomial function $g(y/x, z/x)$ (restricted to $X$) is holomorphic at $p$. More generally, any ratio of homogeneous polynomials in $x, y, z$ (with nonzero denominator at $p$) is holomorphic at $p$.

**Definition 3.18** For $W \subseteq X$ an open set, let $\mathcal{O}_X(W)$ denote the set of holomorphic functions on $W$.

The definitions of removable singularity, pole, and essential singularity on a Riemann surface are completely analogous to the definition of holomorphic function on a Riemann surface. We can determine the kind of singularity $f$ has at a point $p$ by examining its behaviour near $p$:

- If $|f(x)|$ is bounded in a neighborhood of $p$, then there is a removable singularity at $p$ which can be resolved by letting $f(x) = \lim_{x \to p} f(x)$.
- If $|f(x)|$ approaches $\infty$ as $x \to p$, then $p$ is a pole.
- If $|f(x)|$ has no limit as $x \to p$, then $p$ is an essential singularity.

**Definition 3.19 (Meromorphic function)** A function $f$ is meromorphic at $p \in X$ if it is either holomorphic, has a removable singularity, or has a pole, at $p$.

**Example 3.20 (Elliptic function)** Let $\mathbb{C}/L$ be a complex torus with quotient map $\pi : \mathbb{C} \to \mathbb{C}/L$. Let $f$ be a meromorphic function on $\mathbb{C}/L$. The function $g : \mathbb{C} \to \mathbb{C}$ given by $g = f \circ \pi$ is $L-$periodic: $g(z + \omega) = g(z)$ for any $z \in \mathbb{C}$ and $\omega \in L$. Clearly there is a 1-1 correspondence between $L-$periodic functions on $\mathbb{C}$ and functions on $\mathbb{C}$. Further, a meromorphic $L-$periodic function is called an elliptic function and there is a bijection between elliptic functions on $\mathbb{C}$ and meromorphic functions on $\mathbb{C}/L$.

**Definition 3.21** For $W \subseteq X$ an open set, let $\mathcal{M}_X(W)$ denote the set of meromorphic functions on $W$.

Let $f$ be holomorphic on a punctured neighborhood of $p \in X$, with $\phi : U \to V$ a chart containing $p$. So, $f\phi^{-1}$ is holomorphic in a punctured neighborhood of $z_0 = \phi(p)$, so that we may expand $f \circ \phi^{-1}$ in a Laurent series near $z_0$ (with $z$ as the local coordinate in $\mathbb{C}$):

$$f(\phi^{-1}(z)) = \sum_{n=-\infty}^{\infty} c_n(z - z_0)^n.$$

This is the Laurent series for $f$ about $p$ w.r.t $\phi$ (or $z$). The Laurent coefficients depend on the choice of the chart $\phi$.

**Lemma 3.22** *The Laurent series of $f$ about $p$ (w.r.t some chart) has*

- *no negative powers iff $f$ has a removable singularity at $p$,*
- *finitely many negative powers iff $f$ has a pole at $p$,*
- *infinitely many negative powers iff $f$ has an essential singularity at $p$.*

**Definition 3.23 (Order of a meromorphic function at a point)** Let $f$ be meromorphic at $p$ with Laurent series (w.r.t some local variable $z$) $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$, then the order of $f$ at $p$ is

$$\mathrm{ord}_p(f) = \min\{n : a_n \neq 0\}.$$

It has to be checked that $ord_p(f)$ doesn't depend on the choice of chart, which can done by using Lemma 3.3 and composing the power series for the holomorphic transition function. Many of the analogues of the complex analysis theorems carry over.

**Theorem 3.24 (Discreteness of poles and zeroes)** *Let $f$ be a meromorphic function defined in a connected open set $W$ of a Riemann surface $X$. If $X$ is not identically zero, then the the poles and zeroes of $f$ are a discrete subset of $W$.*

Since there can be no limit points in a discrete closed subset, we have:

**Lemma 3.25** *Let $f$ be a meromorphic function on a compact Riemann surface, which is not idenitcally zero. Then $f$ has a finite number of zeroes and poles.*

**Theorem 3.26 (Identity theorem)** *Let $f$ and $g$ be meromorphic on a connected open set $W$ on Riemann surface $X$. Suppose $f = g$ on a subset $S \subset W$ containing a limit point of $W$. Then $f = g$ on $W$.*

**Theorem 3.27 (Maximum Modulus theorem)** *Let $f$ be meromorphic on a connected open set $W$ on Riemann surface $X$. Suppose there is a point $p \in W$ such that $|f(x)| \leq |f(p)|$ for all $x \in W$. Then, $f$ is constant on $W$.*

*Proof.* Suppose $f$ is nonconstant on $W$. Then, $f$ is an open mapping (Open Mapping theorem) and a small disk centered at $p$ is mapped to a disk containing $f(p)$, which would imply existence of $z$ such that $|f(z)| > |f(p)|$. $\square$

Theorem 3.27 holds for harmonic functions too.

**Theorem 3.28 (Analogue of Liouville's theorem)** *If $f$ is holomorphic on a compact Riemann surface $X$, then $f$ is constant.*

*Proof.* Since $|f|$ is continuous and $X$ is compact, $|f|$ achieves a maximum value. Thus, since $X$ is also connected, by Maximum Modulus theorem, $f$ must be constant on $X$. $\square$

## 3.4 Examples of Meromorphic Functions

**Example 3.29 (On Riemann Sphere $\mathbb{C}_\infty$)** Rational functions (ratio of two polynomials) $p(z)/q(z)$ are meromorphic on $\mathbb{C}_\infty$. The converse also holds:

**Theorem 3.30** *Any meromorphic function on the Riemann Sphere is a rational function.*

*Proof.* Let $f$ be a meromorphic function on $\mathbb{C}_\infty$ with poles and zeroes $\{\lambda_i\}$ and $e_i = \mathrm{ord}_{\lambda_i}(f)$ in $\mathbb{C}$. Since $C_\infty$ is compact, $f$ has finite poles and zeroes. Now, let

$$r(z) = \prod_i (z - \lambda_i)^{e_i},$$

be the rational function which has the same poles and zeroes as $f$ in $\mathbb{C}$. Then, consider the meromorphic function $g(z) = f/r(z)$. $g(z)$ has no poles and zeroes in $\mathbb{C}$ and so (possibly after resolving removable singularities) is holomorphic on all of $\mathbb{C}$. Thus, $g(z)$ has a convergent Taylor series:

$$g(z) = \sum_{n=0}^{\infty} a_n z^n.$$

But, $g(z)$ is also meromorphic at $\infty$ in $\mathbb{C}_\infty$, and so

$$g(1/z) = \sum_{n=0}^{\infty} a_n z^{-n},$$

is meromorphic at $z = 0$. For that to be true, the order of $g$ at $0$ needs to be finite, so that $g(z)$ has finitely many terms and is thus a polynomial. If $g(z)$ was nonconstant, then it would have zero(es) in $\mathbb{C}$, which would be a contradiction. So, $f/r$ is constant and thus $f$ is a rational function. $\square$

**Corollary 3.30.1** *Let $f$ be any meromorphic function on $\mathbb{C}_\infty$. Then*

$$\sum_{p\in\mathbb{C}_\infty} \mathrm{ord}_p(f) = 0.$$

**Example 3.31 (On Projective Line)** **Lemma 3.32** *If $p(z, w)$ and $q(z, w)$ are homogeneous of the same degree, with $q not = 0$, then $r(z, w) = p(z, w)/q(z, w)$ descends to a meromorphic function on $\mathbb{P}^1(\mathbb{C})$.*

**Theorem 3.33** *Every meromorphic function on $\mathbb{P}^1(\mathbb{C})$ is a ratio of homogeneous polynomials in $z, w$ of the same degree.*

*Proof.* Let $f$ be meromorphic on $\mathbb{P}^1$. Consider

$$r(z, w) = w^n \prod_i (b_i z - a_i w)^{e_i},$$

where $\{[a_i : b_i]\}$ are the poles and zeros of $f$ and $\mathrm{ord}_{[a_i:b_i]}(f) = e_i$ and $n$ is chosen to homogenize $r$, $n = -\sum_i e_i$. Consider $g = f/r$. Now, $g$ has no poles or zeroes, except possibly at $[1 : 0]$. Argue this can't be a pole and hence $g$ is holomorphic on all of (compact) $\mathbb{P}^1$, so that by (analogue of) Liouville's theorem, $g$ is constant and we're done. $\square$

**Corollary 3.33.1** *Let $f$ be any meromorphic function on $\mathbb{P}^1$. Then*

$$\sum_{p\in\mathbb{P}^1} \mathrm{ord}_p(f) = 0.$$

**Example 3.34 (On Complex Torus)** Fix $\tau$ in upper half-plane $(\mathrm{Im}(\tau) > 0)$, and consider $L = \mathbb{Z} + \tau\mathbb{Z}$. A theta function is

$$\theta(z) = \sum_{n=-\infty}^{\infty} e^{\pi i[n^2\tau+2nz]}.$$

$\theta(z)$ is analytic on all of $\mathbb{C}$. The translate of $\theta(z)$ is

$$\theta^{(x)}(z) = \theta(z - 1/2 - \tau/2 - x).$$

Fix a positive integer $d$ and choose any two sets of $d$ complex numbers $\{x_i\}$ and $\{y_j\}$ such that $\sum_i x_i - \sum_j y_j$ is an integer. Then the ratio of translated theta functions

$$R(z) = \frac{\prod_i \theta^{(x_i)}(z)}{\prod_j \theta^{(y_j)}(z)}$$

is a meromorphic $L$-periodic function on $\mathbb{C}$, and so descends to a meromorphic function on $\mathbb{C}/L$. The converse is also true: any meromorphic function on a complex torus is a ratio of translated theta functions.

**Example 3.35 (On Smooth Affine Plane Curves)** Let $f(x, y) = 0$ define a smooth affine plane curve $X$. Any ratio of two polynomials is meromorphic on $X$ as long as the denominator is not zero everywhere on $X$. This will definitely be the case if $f$ divides the denominator and Hilbert's Nullstellensatz gurantees this is the only case.

**Theorem 3.36 (Hilbert's Nullstellensatz)** *Suppose $h$ is a polynomial vanishing everywhere an irreducible polynomial $f$ vanishes. Then $f$ divides $h$.*

A similar construction works for smooth projective plane curves.

## 3.5 Holomorphic Maps between Riemann Surfaces

Let $X$ and $Y$ be Riemann surfaces.

**Definition 3.37** A mapping $F : X \to Y$ is said to be holomoprhic at $p$ if there exist charts $\phi_1 : U_1 \to V_1$ on $X$ with $p \in U$ and $\phi_2 : U_2 \to V_2$ on $Y$ with $F(p) \in U_2$ such that $\phi_2 \circ F \circ \phi_1^{-1}$ is holomorphic at $\phi_1(p)$.

Again, this definition can be shown to be independent of the choice of charts on $X$ and $Y$.

**Lemma 3.38** *Let $F : X \to Y$ be a holomorphic map, then*

1. *$F$ is continuous and $\mathcal{C}^\infty$.*
2. *If $G : Y \to Z$, $Z$ a Riemann surface, is holomorphic, then $G \circ F : X \to Z$ is holomorphic.*
3. *If $g : W \subseteq Y \to \mathbb{C}$ ($W$ open) is a holomorphic function, then $g \circ F$ is a holomorphic function on $F^{-1}(W)$. In other words, $F$ induces a $\mathbb{C}-$algebra homomorphism:*

$$F^* : \mathcal{O}_Y(W) \to \mathcal{O}_X(F^{-1}(W)),$$

*given by $F^*(g) = g \circ F$.*

4. *If $g : W \subseteq Y \to \mathbb{C}$ ($W$ open) is a mneromorphic function, then $g \circ F$ is a meromorphic function on $F^{-1}(W)$ provided $F(X)$ is not a subset of the set of poles of $g$. In other words, $F$ induces a $\mathbb{C}-$algebra homomorphism:*

$$F^* : \mathcal{M}_Y(W) \to \mathcal{M}_X(F^{-1}(W)),$$

*given by $F^*(g) = g \circ F$.*

**Definition 3.39 (Isomorphism)** An isomorphism between Riemann surfaces $X$ and $Y$ is a bijective holomorphic map $F : X \to Y$ such that $F^{-1} : Y \to X$ is holomorphic. An isomorphism $X \to X$ is an automorphism.

We don't need to check holomorphicity of the inverse if $F$ is 1-1:

**Lemma 3.40** *Let $F : X \to Y$ be an injective holomorphic map between Riemann surfaces. Then $F$ is an isomorphism between $X$ and its image $F(X)$.*

Many of the theorems for holomorphic functions on Riemann surfaces carry over.

**Theorem 3.41 (Open Mapping theorem)** *Let $F : X \to Y$ be a nonconstant holomorphic map. Then $F$ is an open mapping.*

**Theorem 3.42 (Identity theorem)** *Let $F$ and $G$ be two holomorphic maps between $X$ and $Y$ such that $X = Y$ on a subset of $X$ containing a limit point in $X$, then $F = G$ on all of $X$.*

**Theorem 3.43** *Let $X$ be a compact Riemann surface, and let $F : X \to Y$ be a nonconstant holomorphic function. Then $F$ is onto and $Y$ is compact.*

*Proof.* Since $X$ is open and $F$ is nonconstant holomorphic, by the Open Mapping theorem, $F(X)$ is open. Also, since $X$ is compact, $F(X)$ is compact and since $Y$ is Hausdorff, must be closed. Thus, $F(X)$ is closed and open, and since $Y$ is connected, $F(X)$ must be $Y$. □

**Theorem 3.44** *Let $F : X \to Y$ be nonconstant holomorphic. Then for every $y \in Y$, the preimage $F^{-1}(y)$ is a discrete subset of $X$. In particular, if $X$ and $Y$ are compact, then $F^{-1}(y)$ is a nonempty finite set.*

Since any meromorphic function on a Riemann surface can be viewed as a holomorphic map to the Riemann sphere $\mathbb{C}_\infty$ (by letting $f(p) = \infty$ at poles $p$), we have bijection between holomorphic maps $F : X \to \mathbb{C}_\infty$ which are not identically $\infty$ and meromorphic functions $f : X \to \mathbb{C}$.

## 3.6 Global Properties of Holomorphic Maps

Let $X$ and $Y$ be Riemann surfaces with $F : X \to Y$ a holomorphic map.

**Theorem 3.45 (Local Normal Form)** *Let $F$ be nonconstant and defined at $p \in X$. Then there exists a unique integer $m \geq 1$ such that: for every chart $\phi_2 : U_2 \to V_2$ centered at $p$ (i.e., $\phi_2(p) = 0$) there exists a chart $\phi_1 : U_1 \to V_1$ centered at $p$ such that $\phi_2(F(\phi_1^{-1}(z))) = z^m$.*

**Definition 3.46 (Multiplicity)** The multiplicity of $F$ at $p \in X$ is the unique integer $m \in \mathbb{Z}_{\geq 1}$ such that there exists local coordinates near $p$ and $F(p)$ having the form $z \mapsto z^m$.

**Lemma 3.47** *Suppose $z$ is a local coordinate for $p$ so that $z_0$ corresponds to $p$ and $w$ is a local coordinate for $F(p)$ with $w_0$ corresponding to $F(p)$. Then,*

$$\mathrm{mult}_p(F) = 1 + \mathrm{ord}_p(dh/dz),$$

*where $h$ is a holomorphic function such that $w = h(z)$ via $F$. Further, if $h(z) = h(z_0) + \sum_{i=m}^{\infty}(z - z_0)^i$ with $m \geq 1$ and $c_m \neq 0$, then $\mathrm{mult}_p(f) = m$.*

**Definition 3.48 (Ramification point)** Let $F$ be nonconstant. A point $p \in X$ is a ramification point for $F$ if $\mathrm{mult}_p(F) \geq 2$. A point $y \in Y$ is a branch point for $F$ if it is the image of a ramification point of $F$.

**Lemma 3.49** *Let $X$ be a smooth affine plane curve defined by $f(x, y) = 0$. Let $\pi$ be the projection on the first coordinate. Then $\pi$ is ramified at $p \in X$ iff $(\frac{\partial f}{\partial y})(p) = 0$.*

*Let $X$ be a smooth projective plane curve defined by a homogeneous polynomial $F(x, y, z) = 0$; consider $G : X \to \mathbb{P}^1$ defined by $[x : y : z] \mapsto [x, z]$. Then $G$ is ramified at $p \in X$ iff $(\frac{\partial F}{\partial y})(p) = 0$.*

**Lemma 3.50** *Let $f$ be a meromorphic function on a Riemann surface $X$, with a holomorphic map $F : X \to \mathbb{C}_\infty$.*

1. *If $p \in X$ is a zero of $f$, then $\mathrm{mult}_p(F) = \mathrm{ord}_p(f)$.*
2. *If $p$ is a pole of $f$, then $\mathrm{mult}_p(F) = -\mathrm{ord}_p(f)$.*
3. *If $p$ is neither a zero nor a pole of $f$, then $\mathrm{mult}_p(F) = \mathrm{ord}_p(f - f(p))$.*

**Theorem 3.51** *Let $F : X \to Y$ be nonconstant holomorphic map between compact Riemann surfaces. For each $y \in Y$, define $d_y(F)$ to be sum of the multiplicities of $F$ at the points $X$ mapping to $y$:*

$$d_y(F) = \sum_{p \in F^{-1}(y)} \mathrm{mult}_p(F).$$

*Then $d_y(F)$ is constant for all $y \in Y$.*

**Definition 3.52 (Degree of holomorphic map)** Let $F : X \to Y$ be a nonconstant holomorphic map between compact Riemann surfaces. The degree of $F$ $\deg(F)$ is $d_y(F)$.

Thus by Lemma 3.40, we get

**Corollary 3.52.1** *A holomorphic map between compact Riemann surfaces is an isomorphism iff it has degree one.*

Since a meromorphic function on a Riemann surface extends naturally to a holomorphic map to $\mathbb{C}_\infty$, if we have a compact Riemann surface with a meromorphic function having a single pole of order 1, then by the above corollary, it must be isomorphic to the Riemann sphere.

**Theorem 3.53 (#Zeroes = #Poles counted by order)** *Let $f$ be a nonconstant holomorphic function on a compact Riemann surface $X$. Then*

$$\sum_{p \in X} \mathrm{ord}_p(f) = 0.$$

*Proof.* Let $F : X \to \mathbb{C}_\infty$ be the holomorphic map associated to $f$. Then by definition of degree,

$$\deg(F) = \sum_{x_i \text{ a zero of f}} \mathrm{mult}_{x_i}(F) = \sum_{y_j \text{ a pole of f}} \mathrm{mult}_{y_j}(F).$$

Thus by Lemma 3.50, $\sum_{x_i \text{ a zero of f}} \mathrm{ord}_{x_i}(f) = -\sum_{y_j \text{ a pole of f}} \mathrm{ord}_{y_j}(f)$, and the result follows (since any point that is not a pole or zero has order 0). $\square$

**Theorem 3.54 (Meromorphic functions on Complex Torus)** *Any meromorphic function on a complex torus is given by a ratio of translated theta-functions.*

**Theorem 3.55** *For a compact orientable surface without boundary with topological genus $g$, the Euler characteristic is $2 - 2g$.*

**Theorem 3.56 (Hurwitz's Formula)** *Let $F : X \to Y$ be a nonconstant holomorphic function between compact Riemann surfaces. Then*

$$2g(X) - 2 = \deg(F)(2g(Y) - 2) + \sum_{p \in X} [\mathrm{mult}_p(F) - 1].$$

The Hurwitz's formula is counting the Euler characteristic of $X$ in two ways: one using the topological forumula and the other by lifting a triangulation (i.e., good graph) from $Y$ to $X$ via the holomorphic map $F$.

# 4 More examples of Riemann Surfaces

## 4.1 Lines and Conics

**Lemma 4.1** *Any line in $\mathbb{P}^2$ is nonsingular and is isomorphic to $\mathbb{P}^1$.*

*Proof.* A line in $\mathbb{P}^2$ is given by an equation $ax + by + cz = 0$, where not all $a, b, c$ are zero. So a line has atleast one nonzero partial derivative and so is nonsingular. Suppose $c \neq 0$. Then, $[r : s] \mapsto [r : s : -(ar + bs)/c]$ is an isomorphism from $\mathbb{P}^1$ to the line. $\square$

**Definition 4.2 (Conic)** A conic is a quadratic equation of the form

$$F(x, y, z) = ax^2 + 2bxy + 2cxz + dy^2 + 2eyz + fz^2 = 0,$$

where $a, b, c, d, e, f \in \mathbb{C}$ are not all zero. A conic may thus be represented by

$$F(x, y, z) = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = V^\top A_F V,$$

for a symmetric matrix $A_F$.

**Lemma 4.3** *The conic $F$ is nonsingular iff the matrix $A_F$ is invertible.*

*Proof.* The vector of partial derivatives of $F$ is $2A_F V$, $V \in \mathbb{C}^3$. Thus, if $A_F$ is nonsingular, then the null space of $A_F$ is trivial, and so $A_F$ is invertible. Conversely, if $A_F$ is invertible, then $A_F$ has trivial null space and so $F$ is nonsingular. $\square$

**Lemma 4.4** *Let $T$ be an invertible $3 \times 3$ matrix, let $F_A$ be the quadratic equation defined by symmetric matrix $A$, and let $F_B$ be the quadratic equation defined by $B = T^\top AT$. Then the map $T$ that sends $V \mapsto TV$ is an isomorphism from the projective curve $X_B$ defined by $F_B$ to the projective curve $X_A$ defined by $F_A$.*

*Proof.* If the point $V$ lies on $X_B$, then $V^\top(T^\top AT)V = 0$, and so $(TV)^\top A(TV) = 0$, i.e., $TV$ lies on $X_A$. Similarly $T^{-1}$ maps $X_A$ to $X_B$. It needs to be checked that $T$ is holomorphic. $\square$

**Theorem 4.5** *A complex invertible symmetric $A$ can be factored as $A = T^\top T$, for some invertible $T$.*

Thus, we get

**Corollary 4.5.1** *Any smooth projective plane curve is isomorphic to the conic defined by the identity matrix, which is $x^2+y^2+z^2 = 0$. Thus any two smooth projective conics are isomorphic, with an isomorphism $T$ which is an invertible $3 \times 3$ matrix.*

**Lemma 4.6** *Any smooth projective plane conic is isomorphic to $\mathbb{P}^1$ and thus also has topological genus 1.*

*Proof.* Consider the isomorphism mapping $[r : s] \mapsto [r^2 : rs : s^2]$; the inverse maps $[x : y : z]$ to $[x : y]$ or $[y : z]$ depending on whether one of $x, y$ is nonzero or $y, z$ is nonzero. $\square$

## 4.2 Glueing Together Riemann Surfaces

**Theorem 4.7** *Let $X$ and $Y$ be Riemann surfaces. Suppose $U \subseteq X$ and $V \subseteq Y$ are nonempty open sets, and there is given an isomorphism $\phi : U \to V$. Then there is a unique complex structure on the quotient space $Z = X \amalg Y/\phi$ such that the natural inclusions of $X$ and $Y$ are holomorphic. In particular, if $Z$ is a Hausdorff space, then it is a Riemann surface.*

*Proof.* Consider the natural inclusion maps $\iota_X : X \hookrightarrow Z$ and $\iota_Y : Y \hookrightarrow Z$. Then for a chart $\psi : U_X \to \psi(U_X)$, lift it to $Z$ via $\iota_X$ to get a chart on $Z$ $\iota(U_X) \to \iota(\psi(U_X))$ with map $\psi \circ \iota_X^{-1}$.

Similarly, charts from $Y$ can be lifted to $Z$ via $\iota_Y$ to get a complex structure on $Z$. Since $X$ and $Y$ are connected, so is $Z$ and thus it is a Riemann surface iff it is Hausdorff. $\square$

Thus $Z$ is said to be constructed by glueing $X$ and $Y$ along $U$ and $V$ via $\phi$. The Riemann sphere $\mathbb{C}_\infty$ can be constructed by glueing two copies of $\mathbb{C}$ along $\mathbb{C} \setminus \{0\}$ via $z \mapsto 1/z$.

**Definition 4.8 (Hyperelliptic Riemann surfaces)** Let $h(x)$ be a polynomial of degree $2g+1+\epsilon$, where $\epsilon = 0$ or $1$ and assume $h(x)$ has distinct roots. Then let $X$ be the smooth affine plane curve given by $y^2 = h(x)$ with $U \subseteq X = \{(x,y) \in X : x \neq 0\}$. Let $k(z) = z^{2g+2}h(1/z)$ and define $Y$ to be the smooth affine plane curve defined by $w^2 = k(z)$, along with $V \subseteq Y = \{(z,w) \in Y : z \neq 0\}$. Define $\phi : U \to V$ by $(x,y) \mapsto (1/x, y/x^{g+1})$. Then $\phi$ is an isomorphism, and $U$ and $V$ are open sets. Let $Z$ be the Riemann surface obtained by glueing $X$ and $Y$ along $U$ and $V$ via $\phi$. Then we have:

**Theorem 4.9** *$Z$ is a compact Riemann surface with genus $g$. The meromorphic function $x$ on $Z$ (typo in Miranda?) extends to a holomorphic function $\pi : Z \to \mathbb{C}_\infty$, which has degree 2. The branch points of $\pi$ are the roots of $h$ (and the point $\infty$ if $\epsilon = 0$).*

*Proof sketch.* That $Z$ is compact follows from noting that it is a union of the compact sets $\{(x,y) \in X : ||x|| \leq 1\}$ and $\{(z,w) \in Y : ||z|| \leq 1\}$. The map $\pi$ has degree 2 because for a point $x_0 \in \pi(Z)$ such that $h(x_0) \neq 0$, the preimage of $x$ has the points $(x_0, \sqrt{h(x_0)})$ and $(x_0, -\sqrt{h(x_0)})$, each with multiplicity 1.

$Z$ is called a hyperelliptic Riemann surface.

## 4.3 Complex Tori

**Theorem 4.10 (Maps between Complex Tori)** *Let $X$ and $Y$ be two complex tori given by lattices $L$ and $M$. Then any holomorphic map $F : X \to Y$ is induced by a linear map $G : \mathbb{C} \to \mathbb{C}$ of the form $G(z) = \gamma z + a$, for a constant $\gamma$ such that $\gamma L \subseteq M$. The constant $a = 0$ iff $F(0) = 0$; in this case $F$ is a group homomorphism. The holomorphic map $F$ is an isomorphism iff $\gamma L = M$. The degree of $F$ is $[M : \gamma L]$.*

**Theorem 4.11 (Automorphisms of complex tori)** *Let $X = \mathbb{C}/L$ be a complex torus. Then any holormorphic map $F : X \to X$ is induced by multiplication by some $\gamma \in \mathbb{C}$, and so is a group homormorphism. Moreover if $F$ is an automorphism, then either*

1. *$L$ has two orthogonal generators (square lattice), with $\gamma = i$ a generator for $\mathrm{Aut}_0(X)$ (i.e., the automorphisms fixing 0). So $\mathrm{Aut}_0(X) \cong \mathbb{Z}/4$.*
2. *$L$ has two generators at an angle of $\pi/3$ (hexagonal lattice), with $\gamma = e^{\pi/3}$ a generator for $\mathrm{Aut}_0(X)$. So $\mathrm{Aut}_0(X) \cong \mathbb{Z}/6$.*
3. *$L$ is neither square nor hexagonal, and $\gamma = \pm 1$, with $\gamma = \pm 1$. $\mathrm{Aut}_0(X) \cong \mathbb{Z}/2$.*

## 4.4 Groups Actions on Riemann surfaces

A group $G$ is said to act *effectively* on a Riemann surface $X$ if the kernel of its action $K = \{g \in G : gp = p, \text{ for all } p \in X\}$ is trivial. Since $G/K$ always acts effectively, we can assume without loss of generality that $G$ acts effectively. The stabilizer $G_p$ of a point $p$ is also called the *isotropy subgroup* of $p$.

The action of $G$ on $X$ is said to be *continuous* if for every $g$, the mapping $p \mapsto gp$ is a continous map from $X$ to itself. Similarly, the notion of a *holomorphic* action can be defined.

The *quotient space* $X/G$ is the set of orbits with the natural quotient map $\pi : X \to X/G$. We can give this a topology by giving it the quotient topology induced by $\pi$: $U \subseteq X/G$ is open iff $\pi^{-1}(U) \subseteq X$ is open.

The next few theorems are really beautiful for me: a natural blend of algebra and analysis!

**Theorem 4.12** *Let $G$ be a group acting on holomorphically and effectively on Riemann surface $X$. Fix a point $p \in X$. If the stabilizer subgroup $G_p$ is finite, then it is cyclic.*

*Proof.* Fix a local coordinate $z$ centered at $p$. For any $g \in G_p$, consider the Taylor series $g(z) = \sum_{n=1}^{\infty} a_n(g)z^n$, in which $a_0 = 0$ because $g(p) = p$. Also $a_1(g) \neq 0$ because $g$ being an automorphism of $X$ has multiplicity 1 everywhere.

Now the function $a_1 : G_P \to \mathbb{C}^\times$ is a group homomorphism. If we can show it is 1-1, then $a_1(G_p)$ would be finite, and since all finite subgroups of $\mathbb{C}^\times$ are cyclic, we will have the result. So, suppose $g$ is in the kernel of $a_1$, i.e., $g(z) = z +$ higher order terms; we need to show $g(z) = z$. Suppose $m > 2$ is the first nonzero power of $z$ in $g$, so that $g(z) \equiv z + a_m z^m \pmod{z^{m+1}}$, with $a_m \neq 0$. Thus, by induction $g^k(z) \equiv z + k a_m z^m \pmod{z^{m+1}}$. Since $G_p$ is finite, $g$ must have finite order and so $g^k(z) = z$ for some $k$, implying that $a_m = 0$, a contradiction. ☐

**Theorem 4.13** *Let $G$ be a finite group acting holomorphically and effectively on a Riemann surface $X$. Then the points of $X$ with nontrivial stabilizers are discrete.*

*Proof.* Suppose a point $p \in X$ has nontrivial stabilizer but has a sequence of points $\{p_i\}$ converging to it with the same property. Then, for each $p_i$ there is a nontrivial $g_i \in G$ fixing $p_i$. Since $G$ is finite, we can pass to a subsequence $\{p_{i_k}\}$ which is fixed by the same element $g$. Then, since $p_{i_k} \to p$ and $g$ is continuous, $g(p) = p$. Thus, $g$ is the same as the identity function on the subset $\{p_{i_k}\} \cup \{p\}$ containing a limit point, and so by the Idenitity theorem must be the same on all of $X$, contradicting that $G$ acts effectively. ☐

**Theorem 4.14** *Let $G$ be a finite gorup acting holomorphically and effectively on Riemann surface $X$ and fix a point $p \in X$. Then there exists an open set $U \subseteq X$ containing $p$ such that*

1. *$U$ is invariant under the action of $G_p$: for every $g \in G_p, u \in U$, $gu \in U$.*

2. *$U \cap (gU) = \emptyset$ for every $g \notin G_p$.*

3. *The natural map $\alpha : U/G_p \to X/G$ which sends an orbit in $U/G_p$ to its larger orbit in $X/G$ is a homeomorphism onto an open subset of $X/G$.*

4. *No point of $U$ except $p$ is fixed by any nontrivial element of $G_p$.*

*Proof.* Let $\{g_1, \ldots, g_n\} = G \setminus G_p$. Then since $X$ is Hausdorff, we can find disjoint neighborhoods $V_i$ and $W_i$ of $p$ and $g_i p$ respectively. $g_i^{-1} W_i$ is a neighborhood of $p$ for each $i$. Let $R_i = V_i \cap (g_i^{-1} W_i)$, $R = \bigcap_i R_i$, and $U = \bigcap_{g \in G_p} gR$. $U$ satisfies the conditions. The last one follows from the fact that points with nontrivial isotropy are discrete. $\square$

This proposition is key to giving $X/G$ a complex structure. For a point $p$ with trivial isotropy, we can define a chart in the following way. By the above proposition we can find a neighborhood $U$ of $p$ which is $G_p$-invariant and $\pi_{|U} : U \to W \subseteq X/G$ is a homeomorphism. By shrinking $U$ if necessary, we can assume $U$ is the domain of a chart $\phi : U \to V \subseteq \mathbb{C}$ on $X$. Thus, the composition $\psi : \phi \circ \pi_{|U}^{-1} : W \to V$ is a chart on $X/G$.

If $p \in X$ has nontrivial isotropy, i.e., $m = |G_p| \geq 2$, then we have to do a little more work to extract a chart on $X$ that descends to a chart on $X/G$. Again, start with a neighborhood $U$ containing $p$ that is $G_p$-invariant that is $m$-to-1 away from $p$ (by point 4 of the previous thm). So we seek a homeomorphism $h$ satisfying

$$h : U \to U/G_p \xrightarrow{\alpha} W \xrightarrow{\phi} \mathbb{C}$$

This means $h$ has to be $G_p$-invariant. Fix a local coordinate centered at $p$, say $z$. Let

$$h(z) = \prod_{g \in G_p} g(z).$$

Since each $g \in G_p$ has multiplicity 1 at $p$, $h$ has multiplicity $m$ at $p$. By suitably shrinking $U$, we may also assume $h$ is defined on $U$. Also, $h$ is $G_p$-invariant since $h(gz) = g(z)$ for any $g \in G_p$. Thus, we get a well-defined map $\overline{h} : U/G_p \to \mathbb{C}$ that is continuous and open. Also $\overline{h}$ is injective since the projection $U \to U/G_p$ is $m$-to-1 away from $p$ and the map $h$ having multiplicity $m$ is also $m$-to-1 away from $p$ (also $\alpha$ and $\phi$ are injective).

Therefore, we can let $\phi$ be

$$\phi = \overline{h} \circ \alpha^{-1} : W \to \mathbb{C}.$$

These charts make $X/G$ into a Riemann surface and the thing remaining to be checked is that the charts defined are compatible. The only case we have to really worry about is the nonempty intersection of the domain of two points, one which has nontrivial isotropy and the other has trivial isotropy (we don't have to check both having nontrivial stabilizers because we know by a preceding theorem that such points are discrete). This is not hard to check.

By the definition of the charts on $X/G$, we know that $\mathrm{mult}_p(\pi) = |G_p|$. So, we can also notice that the degree of the projection map $\phi : X \to X/G$ exists and is equal to $|G|$ since, by the Orbit-Stabilizer theorem (or the fact the points of the same orbit have conjugate stabilizers), $\sum_{x\in\pi^{-1}(\mathcal{O}_p)} \mathrm{mult}_x(\pi) = \sum_{x\in\pi^{-1}(\mathcal{O}_p)} |G_x| = \sum_{x\in\pi^{-1}(\mathcal{O}_p)} |G_p| = |G|$.

We also have a linearization of action:

**Theorem 4.15 (Linearization of the Action)** *Let $G$ be a finite group acting effectively and holomorphically on a Riemann surface $X$. Fix a point $p \in X$ with nontrivial isotropy of order $m$. Let $g \in G_p$ generate $G_p$. Then there is a local coordinate $z$ on $X$ centered at $p$ such that $g(z) = \lambda z$, where $\lambda$ is a primitive $m^{th}$ root of unity.*

*Proof.* Fix a local coordinate $w$ near $\mathcal{O}_p$. By the normal local form, we can find a local coordinate $z$ near $p$ such that $w = z^m$. For nonzero values of $w$ near 0, the corresponding preimages are all off by an $m$-th root of unity and so form the set $\{e^{2\pi ik/m} : 0 \leq k \leq m-1\}$. Since $G_p$ locally acts on $X$ near $p$, these points form a $G_p$-orbit. Thus, $g(z) = e^{2\pi ik/m}z$ for some $1 \leq k \leq m-1$ ($k \neq 0$ since $G_p$ is assumed to be nontrivial). □

Next, we want to understand the ramification of the projection map $\pi$. Let $X$ be a compact Riemann surface. Suppose $y \in X/G$ is a branch point of $\pi$; let the points lying above $y$ in $X$ be $\{x_1, x_2, \ldots, x_s\}$. These points $x_i$ all belong to the same orbit and thus have the same isotropy, say $r$. Then, by the orbit stabilizer thm we have that $s = |G|/r$. Thus, if $\pi$ has branch points $y_1, \ldots, y_k$, with $\pi$ having multiplicity $r_i$ at the points above the branch points, then by Riemann-Hurwitz formula, we have

$$2g(X) - 2 = |G|\left(2g(X/G) - 2\right) + \sum_{j=1}^{k} \frac{|G|}{r_j}(r_j - 1).$$

Using this we can say a lot about the possible ramification indexes given the genus of $X$ (Miranda pp. 80-81). We also get

**Theorem 4.16 (Hurwitz's theorem)** *Let $G$ be a finite group acting effectively and holomorphically on a compact Riemann surface $X$ with genus $g \geq 2$. Then*

$$|G| \leq 84(g-1).$$

# 5 Monodromy

## 5.1 Fundamental group

Given two paths $\gamma_1$ and $\gamma_2$ in a space $X$, we defined in Definition 1.8 what it means for two paths to be homotopic. We can show that homotopy of paths with common endpoints is an equivalence relation. In particular, we can look at all the homotopy classes of loops based at a fixed point in $X$ and show that this set has a group structure. The operation in for this group is the product or concatenation of paths: given two loops $\gamma, \beta : [0,1] \to X$ based at $q \in X$, we define $\gamma \cdot \beta$ as the path $\gamma(2s)$ for $0 \leq s \leq 1/2$ and $\beta(2s-1)$ for $1/2 \leq s \leq 1$. (This product works for more general paths with the constraint that the endpoint of $\gamma$ and the start point of $\beta$ coincide: $\gamma(1) = \beta(0)$.) Thus, without too much trouble it can be shown that homotopy classes of loops based at $q$ form a group denoted $\pi_1(X, q)$ called the **fundamental group**.

We also have the useful van Kampen theorem that, for instance, implies that the fundamental group of a point with $m$ loops based at it is the free group of $m$ generators.

## 5.2 Covering Space

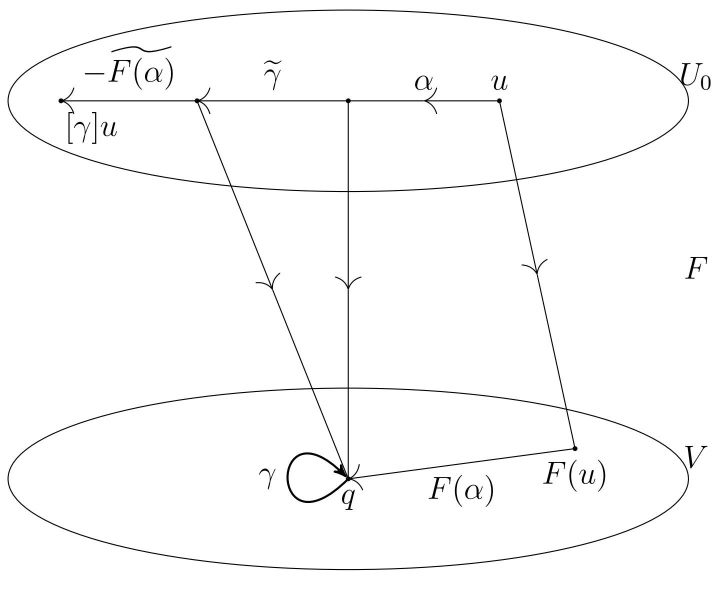
A **covering space** of a space $V$ is a continuous surjective map $F : U \to V$ such that for each $v \in V$ there exists a neighborhood $W \subseteq V$ of $v$ so that

$$F^{-1}(W) = \bigsqcup_{u \in F^{-1}(v)} N_u,$$

where $N_u \subseteq U$ is a neighborhood of $u$ for which $F : N_u \to W$ is a homeomorphism.

In particular, if $F_0 : U_0 \to V$ is a covering space and $U_0$ is simply connected, then this is called a **universal covering space** of $V$. The existence of a universal covering space is guaranteed. The universal covering of $V$ is unique upto isomorphism and its universal property is any covering space $F : U \to V$ factors through $F_0$.

**Definition 5.1 (Action of $\pi_1$ on universal covering)** Given a space $V$ and a fixed base-point $q \in V$, we define an action of the fundamental group at $q$ on the universal covering $F : U_0 \to V$ in the following way:



The action of $\pi_1$ collapses the fiber above each point in $V$ to a single point, so that the orbit space $U_0/\pi_1(X,q)$ is homeomorphic to $V$. We can also act on the universal covering by a subgroup $H \leq \pi_1(X,q)$, in which case we can show that $U_0/H$ is a covering space of $V$. Furthermore, every covering space of $V$ arises in this way.

### 5.3 Monodromy representation of a finite degree covering

We start with a covering space $F : U \to V$ and suppose that $F$ is of finite degree $d$, i.e., $U$ is a $d$-sheeted covering of $V$ and $U$ is connected.

Fix a base-point $q \in V$ and a loop $\gamma$ based at $q$. Then lift $\gamma$ to $U$: suppose the fiber above $q$ is $\{x_1, \ldots, x_d\}$. So for each $x_i$ there is a unique lift $\tilde{\gamma}_i$ such that $\tilde{\gamma}_i(0) = x_i$. Now for each $i$, the endpoint $\tilde{\gamma}_i(1) = x_j$ for some $1 \leq j \leq d$. By the uniqueness of lifts for coverings, we can show the association of endpoints $i \mapsto j$ is a permutation of $\{1, \ldots, d\}$. Thus, we have a homomorphism $\rho : \pi_1(V,q) \to S_d$ and this is called the **monodromy representation** of $F : U \to V$.

## 6 Hurwitz Numbers

### 6.1 Riemann's Existence theorem

$Y$ is a compact Riemann surface. The motivating question is to understand the different holomorphic maps to $Y$ when branch points in $Y$ specified. Naturally, we have to talk about when two holomorphic maps to $Y$ are considered equivalent.

**Definition 6.1 (Isomorphism of Holomorphic maps)** Let $f : X \to Y$ and $g : \tilde{X} \to Y$ be holomorphic maps. Then $f$ and $g$ are called isomorphic if there is an isomorphism $\phi : X \to \tilde{X}$ such that $f = g \circ \phi$, and $\phi$ is called an isomorphism of $f$. When $X = \tilde{X}$, $\phi$ is called an automorphism of $f$ and the group of automorphisms of $f$ is $\mathrm{Aut}(f)$.

To talk about the combinatorics of ramification numbers, we keep in mind the notion of a **partition**: a partition of a positive integer $d$ is an *unordered* $k$-tuple $(a_1, \ldots, a_k)$ such that $\sum_i a_i = d$; if $a_i = a_j$ for some $i \neq j$, we still think of them as distinct elements of the partition. This is so that we can talk about functions from a partition; an automorphism of a partition is a bijection that only permutes among numerically equal elements. The **size** of a partition is $d$ and the **length** is the number of elements in the tuple.

Let $f : X \to Y$ be holomorphic and fix a $y \in Y$. Then, if $f^{-1}(y) = \{x_1, \ldots, x_k\}$, for each $i$ there is a local normal form near $x_i$ such that $w$ and $z_i$ are local coordinates near $y$ and $x_i$, then $w = z_i^{r_i}$ (i.e., $r_i$ is the multiplicity of $x_i$ in $f$). The tuple $(r_1, r_2, \ldots, r_k)$ is called the **ramification profile** of $f$ at $y$. If $d$ is the degree of the $f$, then the ramification profile is a partition of $d$. When

$k = d$, so that $r_i = 1$ for all $i$, $f$ is **unramified** at $y$, when $k = 1$, $f$ is **fully ramified** at $y$, and if $k = d - 1$ (so $(2, 1, \ldots, 1)$), $f$ is **simply ramified** at $y$.

**Definition 6.2 (Hurwitz Numbers)** Let $Y$ be a compact Riemann surface with genus $g$. For a positive integer $d$, let $\lambda_1, \ldots, \lambda_n$ be paritions of $d$. Then,

$$H_{h \xrightarrow{d} g}(\lambda_1, \ldots, \lambda_n) = \sum_{[f]} \frac{1}{|\mathrm{Aut}(f)|},$$

where the sum is over all isomorphism classes of maps $f$ such that

- $f : X \to Y$ is a holomorphic map of degree $d$ from a compact (connected) Riemann surface $X$ of genus $h$,
- The branch points of $Y$ in $f$ are $\{b_1, \ldots, b_n\}$ such that $b_i$ has ramification profile $\lambda_i$.

Any map $f$ from the above definition is called a **Hurwitz cover** for the parameters $g, h, d, \lambda_1, \ldots, \lambda_n$.

For example $H_{0 \xrightarrow{d} 0}$ is $1/d$ since every map between Riemann spheres can be shown to be isomorphic to the power map $p(x) = x^d$ and $\mathrm{Aut}(p) \cong \mu_d$, the multiplicative group of $d$th roots of unity.

Since unlike Miranda, Cavalieri and Miles don't require Riemann surfaces to be connected, they also consider the similarly defined notion of a Hurwitz cover arising from a disconnected Riemann surface, by letting the genus of such a surface being determined by its Euler characteristic so that is true $\chi(X) = 2 - 2g$. Therefore, if a disconnected Riemann surface $X$ is the union of $n$ connected Riemann surface with genera $g_i$, then because $\chi$ is additive for unions of disjoint surfaces,

$$g(X) = g_1 + \cdots + g_n + 1 - n.$$

They denote the Hurwitz number for covers coming from a disconnected Riemann surface by $H^{\bullet}_{h \xrightarrow{d} g}(\lambda_1, \ldots, \lambda_n)$.

We already know that holomorphic maps between Riemann surfaces are ramified coverings, giving true coverings away from ramification points and branch points. The Riemann Existence theorem says the converse it also true: every ramified covering of a compact Riemann surface is a holomorphic map.

**Theorem 6.3 (Riemann's Existence theorem)** *Suppose $Y$ is a compact Riemann surface and $\tilde{f} : \tilde{X} \to Y \setminus \{b_1, \ldots, b_n\}$ is a covering from a topological surface $\tilde{X}$. Then $\tilde{X}$ can be uniquely extended to a compact Riemann surface such that $\tilde{f}$ extends to a holomorphic map $f : X \to Y$.*

*Proof.* [Sketch] The idea is to make maximum use of the assumption that $\tilde{f} : \tilde{X} \to Y$ is a covering. For each branch point $b_i$, we choose a chart $\varphi_i$ centered at $b_i$, and take $V_i$ to be the open neighborhood $\varphi_i^{-1}\{|w| < 1\}$ of $b_i$. Then, since $\tilde{f}$ is a covering of say degree $m$, $\tilde{f}^{-1}(V_i \setminus \{b_i\})$ will be the disjoint union of some open neighborhoods $\tilde{U}_1, \ldots, \tilde{U}_m$. So, since $\tilde{U}_i$ is homeomorphic to $V_i \setminus \{b_i\}$, which in turn is homeomorphic to the complex unit punctured disk, we have a homeomorphism $\phi_i : \tilde{U}_i \to D \setminus \{0\}$ ($D$ denoting unit disk). Then, we can extend $\phi_i$ to ahomeomorphism $\tilde{U}_i \cup \{x_i\} \to D$ by adding a point $x_i$ to $\tilde{X}$ (I think, explicitly we could let $x_i = \lim_{w\to 0} \phi_i^{-1}(w)$). Thus, $X = \tilde{X} \cup \{x_1, \ldots, x_m\}$ is the compact surface we want. The complex structure of $X$ is just lifted from $Y$: for each $x \in \tilde{x}$, we have an open neighborhood $U_x$ such that $\tilde{f} : U_x \to f(U_x)$ is a homeomorphism; so taking a chart $\varphi_x$ containing $f(U_x)$, we will have $\varphi_x \circ f$ serve as a chart at $x$ in the neighborhood $U_x$. For the points $x_i$, the homeomorphisms $\varphi_i$ considered before serve as charts. Moreover, this complex structure ensures that $f$ is holomorphic simply because in the local coordinates of these charts, $f$ is just the identity. $\square$

## 6.2 Hyperelliptic Covers

In Miranda, the glueing construction of a hyperelliptic Riemann surface showed that the projection map to the Riemann sphere had degree 2. In Cavalieri and Miles, this is taken as the definition of a hyperelliptic Riemann surface. A map of degree 2 from a hyperelliptic Riemann surface to $\mathbb{P}^1$ is called a **hyperelliptic cover**. By the Riemann-Hurwitz formula, we see there have to be $2g + 2$ ramification points of multiplicity 2, and so there are $2g + 2$ branch points of a hyperelliptic cover. They show that upto isomophism there is precisely one hyperelliptic cover from each compact Riemann surface of genus $g$ and that this map has just one nontrivial isomorphism, implying

$$H_{g \xrightarrow{2} 0}((2)^{2g+2}) = \frac{1}{2}.$$

First of all, they show that there is atleast one hyperelliptic cover originating from a genus $g$ Riemann surface and their explicit construction is the one given in Miranda (actually they don't consider the glueing construction, but appeal to the Riemann Existence theorem to extend the affine plane curve to the required surface)! After some thought I realized this should be the case, otherwise taking into account that there is just one hyperelliptic cover, we would have two different definitions of a hyperelliptic Riemann surface.

## 6.3 Counting Monodromy Representations

Cavalieri and Miles introduce the same notion of monodromy representations as Miranda. They connect this to the partitions of degree $d$: since permutations in $S_d$ have the same cycle type if and only if they are conjugates, we can index conjugacy classes by partitions of $d$.