REPRESENTATION THEORY OF HEISENBERG GROUP

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The Heisenberg group is $\mathrm{Heis}(q) = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{F}_q \right\}$.

1. Conjugacy classes

Let $g = \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$. We have the following calculation:

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & a & c + (bx - ay) \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}.$$

So, there are two types of classes depending on whether $(a, b) = (0, 0)$.

- $(a, b) = (0, 0)$. Then it is clear that g is fixed under conjugates, so there are q conjugacy classes of single elements.
- $(a, b) \neq (0, 0)$. Fix $d \in \mathbb{F}_q$. We need to check how many solutions $(x, y) \in \mathbb{F}_q^2$ exist to
$$bx - ay = d.$$
Clearly, there are q solutions. So, there are $(q^3 - q)/q = q^2 - 1$ classes of size q each in this case.

2. Representations of Heis(q)

- q^2 irreducible representations of dimension 1: for each $(a, b) \in \mathbb{F}_q^2$,
$$\mu_{a,b} \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \psi(ax + by),$$
where $\psi(x) = \exp(\frac{2\pi i \operatorname{Tr}(x)}{p})$. Here we are taking $q = p^n$ and the trace is
$$\operatorname{Tr}(x) = x + x^p + \cdots + x^{p^{n-1}}.$$
- $q - 1$ irreducible representations of dimension q: for each $s \in \mathbb{F}_q^\times$, let $A \subseteq G = \mathrm{Heis}(q)$ be the subgroup of elements with $x = 0$. Then
$$\pi_s = \operatorname{Ind}_A^G \psi_s,$$
where
$$\psi_s \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \psi(sz).$$

Since ψ maps to the complex unit circle, it is easy to check these are all indeed irreducible and similarly simple to check they are inequivalent.

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3. Dimension q representations

We can now try working out π_s explicitly following Terras' Proposition 1 in Ch. 16. A useful obervation is that

$$\begin{bmatrix}1 & x & z\\0 & 1 & y\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & z\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & y\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & x & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}.$$

So, it suffices to define π_s on each of these three types of matrices. From the above decomposition, we also see that a complete set of unique representatives for $A \setminus G$ is

$$\left\{\begin{bmatrix}1 & x & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} : x \in \mathbb{F}_q\right\}.$$

We denote these representatives by g_i.

Now, an orthonormal basis for the vector space V of functions $f : G \to \mathbb{C}$ such that $f(hg) = \psi_s(h)f(g)$ for all $h \in A$ and $g \in G$, is

$$f_i(g) = \psi_s(gg_i^{-1})\delta_A(gg_i^{-1}),$$

where δ_A is the delta function that is supported exactly on A.

- Let $g = \begin{bmatrix}1 & 0 & z\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$. Then for any $t = \begin{bmatrix}1 & a & c\\0 & 1 & b\\0 & 0 & 1\end{bmatrix} \in G$,

$$\begin{aligned}\pi_s g(f_i)(t) &= f_i(tg)\\ &= \psi_s(tgg_i^{-1})\delta_A(tgg_i^{-1}).\end{aligned}$$

  Now,

$$tgg_i^{-1} = \begin{bmatrix}1 & a & c\\0 & 1 & b\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & z\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & -x_i & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & a-x_i & z+c\\0 & 1 & b\\0 & 0 & 1\end{bmatrix},$$

  and so $\delta_A(tgg_i^{-1}) = 1$ precisely when $a = x_i$, in which case

$$\pi_s g(f_i)(t) = \psi(s(z+c)) = \psi(sz)f_i(t).$$

  So, we have worked out that

$$\pi_s \begin{bmatrix}1 & 0 & z\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \psi(sz)I_q,$$

  where I_q is the identity matrix of size q.

- Let $g = \begin{bmatrix}1 & 0 & 0\\0 & 1 & y\\0 & 0 & 1\end{bmatrix}$. In this case,

$$tgg_i^{-1} = \begin{bmatrix}1 & a & c\\0 & 1 & b\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & y\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & -x_i & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & a-x_i & ay+c\\0 & 1 & y+b\\0 & 0 & 1\end{bmatrix},$$

  and so again $\delta_A(tgg_i^{-1}) = 1$ precisely when $a = x_i$, in which case

$$\pi_s g(f_i)(t) = \psi(s(x_i y + c)) = \psi(sx_i y)f_i(t)$$

So,

$$\pi_s \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \psi(sx_1y) & 0 & \dots & 0 \\ 0 & \psi(sx_2y) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \psi(sx_qy) \end{bmatrix},$$

where $\mathbb{F}_q = \{x_1, \dots, x_q\}$.

- Let $g = \begin{bmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Here

$$tgg_i^{-1} = \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -x_i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+x-x_i & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix},$$

and $\delta_A(tgg_i^{-1}) = 1$ when $a = x_i - x$, in which case

$$\pi_s g(f_i)(t) = \psi(sc) = f_{(x_i-x)_j}(t).$$

Thus, $\pi_s \begin{bmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the permutation matrix $\delta_{x_i-x}(a)$.

3.1. **Character.** We can also calculate the character χ_{π_s} using the Frobenius formula (Terras pp. 271):

$$\chi_{\pi_s}(g) = \sum_{g_i \in A \backslash G} \chi_{\psi_s}(g_i g g_i^{-1}) \delta_A(g_i g g_i^{-1}).$$

When $g = (a, b, c)$ is such that $a \neq 0$, then $\delta_A(g_i g g_i^{-1}) = 0$. So, suppose $a = 0$. Then,

$$\begin{aligned} \chi_{\pi_s}(g) &= \sum_{x \in \mathbb{F}_q} \psi(s(c+bx)) \\ &= \psi(sc) \sum_{x \in \mathbb{F}_q} \psi(sbx) \\ &= \psi(sc) \sum_{x \in \mathbb{F}_q} e^{2\pi i \operatorname{Tr}(x)/p}, \end{aligned}$$

where in the last equality we are assuming $b \neq 0$, because otherwise the sum is simply $q\psi(sc)$. We need to understand $\operatorname{Tr}(x)$. Intuitively, it should be equidistributed over $\mathbb{F}_p$. This can be proved as follows (are there simpler ways?). The map $\operatorname{Tr} : \mathbb{F}_q \to \mathbb{F}_p$ has kernel exactly $\{x^p - x : x \in \mathbb{F}_q\}$ (by Hilbert's theorem 90). But the map $H : x \mapsto x^p - x$ has kernel precisely $\mathbb{F}_p$, and so $|\ker(\operatorname{Tr})| = |\operatorname{im}(H)| = p^n/p = p^{n-1}$. Therefore, $|\operatorname{im}(\operatorname{Tr})| = p^n/p^{n-1} = p$. Thus, Tr is surjective, which paired with the additivity and kernel size p^{n-1} of Tr easily implies that Tr is indeed equidistributed. So, if $b \neq 0$, then the above sum is

$$= \psi(sc) p^{n-1} \sum_{x=0}^{p-1} e^{2\pi i x/p} = 0.$$

So, $\chi_{\pi_s}(0, 0, c) = q\psi(sc)$ and it is zero on the other conjugacy classes.

4. Ramanujan graphs

A connected $(q+1)$-regular graph X is *Ramanujan* if

$$\mu := \max\{|\lambda| : \lambda \in \mathrm{Spec}(X), |\lambda| \neq q+1\},$$

is such that $\mu \leq 2\sqrt{q}$.

We can use the Cayley graphs of Heis(p) to give a few examples of Ramanujan graphs. We will denote elements of $G = \mathrm{Heis}(p)$ by tuples (x, y, z). Let $S = \{(x, y, 0) \in \mathbb{F}_p^3 : \text{exactly one of } x \text{ or } y = 0\ \}$. Then, it is easy to check that S is a symmetric generating set for Heis(p). To check whether the Caylely graph $X = X(G, S)$ is a Ramanujan graph, we calculate the spectrum of X using its irreducible representations. We have

$$A(X) \simeq \bigoplus_{\pi \in \widehat{G}} d_\pi M_\pi,$$

where

$$M_\pi = \sum_{s \in S} \pi(s).$$

Note that X is $2(p-1)$-regular.

For $(a, b) \in \mathbb{F}_p^2$ and $\pi = \mu_{a,b}$, we have

$$\begin{aligned} M_\pi &= \sum_{x=1}^{p-1} (\mu_{a,b}(x,0,0) + \mu_{a,b}(0,x,0)) \\ &= \sum_{x=1}^{p-1} (e^{2\pi i a x/p} + e^{2\pi i b x/p}) \\ &= \begin{cases} -2 & ab \neq 0 \\ 2(p-1) & (a,b) = (0,0) \\ p-2 & \text{else.} \end{cases} \end{aligned}$$

But $p - 2 \leq 2\sqrt{2p-3}$ only if $p = 2, 3, 5, 7$.

Next, for $s \in \mathbb{F}_p^\times$ and $\pi = \pi_s$, we have

$$\begin{aligned} M_\pi &= \sum_{x=1}^{p-1} (\pi_s(x,0,0) + \pi_s(0,x,0)) \\ &= \sum_{x=1}^{p-1} P^x + \sum_{x=1}^{p-1} D(sx) \\ &= \begin{bmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 0 \end{bmatrix} + \begin{bmatrix} p-1 & 0 & \dots & 0 \\ 0 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 \end{bmatrix} \\ &= \begin{bmatrix} p-1 & 1 & \dots & 1 \\ 1 & -1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & -1 \end{bmatrix}. \end{aligned}$$

Here P is a primitive permutation matrix and $D(sx)$ is a diagonal matrix $(\psi(sxn))_{0\leq n\leq p-1}$. The eigenvalues of M_π can be calculated to be -2 (multiplicity $p-2$) and $p-2\pm\sqrt{p}$. This prevents $p=7$ from being Ramanujan too. So, Heis(p) for $p=2,3,5$ give us Ramanujan graphs.