REPRESENTATION THEORY OF $\mathrm{GL}_2(\mathbb{F}_q)$

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The group $\mathrm{GL}_2(\mathbb{F}_q) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{F}_q, ad - bc \neq 0 \right\}$. We have

$$|\mathrm{GL}_2(\mathbb{F}_q)| = q(q+1)(q-1)^2.$$

A nice way to see this is to consider the transitive action of $\mathrm{GL}_2(\mathbb{F}_q)$ on $\mathbb{P}^1(\mathbb{F}_q)$. Then

$$\mathrm{stab}((1:0)) = B := \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \right\}.$$

So,

$$|\mathrm{GL}_2(\mathbb{F}_q)| = |\mathbb{P}^1(\mathbb{F}_q)| \cdot |B| = (q+1) \cdot q(q-1)^2.$$

1. Conjugacy classes

The conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_q)$ are determined by Jordan normal forms.

- Central: $g = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$. Such an element is stable under conjugation, so there is a single element in this class and $q-1$ such classes.
- Parabolic: $g = \begin{bmatrix} r & 1 \\ 0 & r \end{bmatrix}$. We have

  $$\mathrm{stab}(g) = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \right\},$$

  so there are $|\mathrm{GL}_2(\mathbb{F}_q)|/q(q-1) = q^2 - 1$ elements in the class and $q-1$ such classes.
- Hyperbolic: $g = \begin{bmatrix} r & 1 \\ 0 & s \end{bmatrix}, r \neq s$. We get

  $$\mathrm{stab}(g) = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \right\},$$

  and thus there are $|\mathrm{GL}_2(\mathbb{F}_q)|/(q-1)^2 = q(q+1)$ elements in the class and $(q-1)(q-2)/2$ such classes (we divide by 2 because the order of r and s does not matter).
- Elliptic: Fix an element $\delta \in \mathbb{F}_q \setminus \mathbb{F}_q^2$. Then $g = \begin{bmatrix} r & s\delta \\ s & r \end{bmatrix}, s \neq 0$. We see

  $$\mathrm{stab}(g) = \left\{ \begin{bmatrix} a & b\delta \\ b & a \end{bmatrix} \right\} \simeq \mathbb{F}_q(\sqrt{\delta})^\times \simeq \mathbb{F}_{q^2}^\times,$$

  and so there are $|\mathrm{GL}_2(\mathbb{F}_q)|/(q^2-1) = q(q-1)$ elements in the class and $q(q-1)/2$ such classes (here we divide by 2 because the sign of s does not matter).

In all, there are $q^2 - 1$ conjugacy classes.

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2. Representations of $\mathrm{GL}_2(\mathbb{F}_q)$

2.1. **Type I.** These irreducible representations are induced by characters of the subgroup of upper triangular matrices B.

- Given a character α of $\mathbb{F}_q^\times$, we get a degree-1 representation of $\mathrm{GL}_2(\mathbb{F}_q)$ by composing α with the $\det : \mathrm{GL}_2(\mathbb{F}_q) \to \mathbb{F}_q^\times$ map. There are $q-1$ of these by the duality of abelian groups.
- Principal series representation: Let $\pi = \mathrm{Ind}_B^G 1$. Unfortunately, π is not quite irreducible since it has the trivial representation in it. But by removing the trivial representation, we get π_0, an irreducible representation, i.e.,

$$\pi = \pi_0 \oplus 1.$$

  For each 1-d representation α, we obtain a $(q+1)$-degree irreducible representation by tensoring:

$$\pi_\alpha = \pi_0 \otimes \alpha.$$

  There are $q-1$ of these. Let us now calculate the character χ_{π_α} using the Frobenius formula with modifications to account for removing the trivial representation and tensoring with α,

$$\chi_{\pi_\alpha}(g) = \left(-1 + \sum_{g_i \in B \setminus G} \delta_B(g_i g g_i^{-1}) \right) \alpha(\det(g)).$$

  A set of representatives for $B \setminus G$ is

$$\left\{ \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} : x \in \mathbb{F}_q \right\} \cup \{I\},$$

  where I is the identity matrix.

  - Central g:
$$\chi_{\pi_\alpha}(g) = (-1 + (q+1))\, \alpha(r^2) = q\alpha(r)^2.$$
  - Parabolic g:
$$\chi_{\pi_\alpha}(g) = (-1+1)\alpha(r^2) = 0.$$
  - Hyperbolic g:
$$\chi_{\pi_\alpha}(g) = (-1 + (q+1))\, \alpha(rs) = q\alpha(rs).$$
  - Elliptic g:
$$\chi_{\pi_\alpha}(g) = (-1+0)\, \alpha(r^2 - s^2\delta) = -\alpha(N(z)),$$
    where $N : \mathbb{F}_q(\sqrt{\delta})^\times \to \mathbb{F}_q^\times$ is the norm map, and we are denoting $z = r + s\sqrt{\delta}$.

- Given two characters $\alpha \neq \beta$ of $\mathbb{F}_q^\times$, we define a 1-d representation $\mu_{\alpha,\beta}$ of B by

$$\mu_{\alpha,\beta} \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \alpha(a)\beta(d).$$

  We then get an irreducible $(q+1)$-degree representation of G by induction

$$\rho_{\alpha,\beta} = \mathrm{Ind}_B^G \mu_{\alpha,\beta}.$$

  Since $\rho_{\alpha,\beta} \simeq \rho_{\beta_\alpha}$, we have $(q-1)(q-2)/2$ of these. The character is given by

$$\chi_{\rho_{\alpha,\beta}}(g) = \sum_{g_i \in B \setminus G} \delta_B(g_i g g_i^{-1}) \mu_{\alpha,\beta}(g_i g g_i^{-1}).$$

  - Central g:
$$\chi_{\rho_{\alpha,\beta}}(g) = (q+1)\alpha(r)\beta(r).$$

– Parabolic g:
$$\chi_{\rho_{\alpha,\beta}}(g) = \alpha(r)\beta(r),$$
where this term comes from $g_i = I$ and the other terms are 0.

– Hyperbolic g:
$$\chi_{\rho_{\alpha,\beta}}(g) = q\alpha(s)\beta(r) + \alpha(r)\beta(s).$$
Note: this value does not match what Terras has given. I am not sure why – maybe there is a problem with the coset representatives I have chosen but all the other calculations work out fine.

– Elliptic g:
$$\chi_{\rho_{\alpha,\beta}}(g) = 0.$$

2.2. **Type II.** There are called Discrete series representation or cuspidial representations. Fix a character ν of $\mathbb{F}_q(\sqrt{\delta})^\times$. We have to assume ν is *nondecomposable*: there does *not* exist a character χ of $\mathbb{F}_q^\times$ such that
$$\nu = \chi \circ N_{\mathbb{F}_q(\sqrt{\delta})/\mathbb{F}_q}.$$
Since $\mathbb{F}_q(\sqrt{\delta}) \simeq \mathbb{F}_{q^2}$, the norm map is simply $N : z \mapsto z \cdot z^q$. The nondecomposibility is, in fact, equivalent to $\nu \neq \nu^q$. Indeed, if ν were nondecomposable, then
$$\nu(z)^q = \nu(z^q) = \chi(N(z^q)) = \chi(N(z)) = \nu(z).$$
For the converse, suppose $\nu = \nu^q$.. Let $\chi : \mathbb{F}_q^\times \to \mathbb{C}^\times$ be $\chi(x) = \sqrt{\nu(x)}$. Then we claim $\nu = \chi \circ N$. Indeed, let $z \in \mathbb{F}_{q^2}^\times$ with $N(z) = a \in \mathbb{F}_q^\times$. Then, since $\nu(z) = \nu(z)^q$,
$$\nu(z)^2 = \nu(z)^{q+1} = \nu(z^{q+1}) = \nu(a),$$
and so
$$\nu(z) = \sqrt{\nu(a)} = \chi(a) = \chi(N(z)).$$
A consequence of this is that
$$\sum_{N(z)=1} \nu(z) = 0. \tag{2.1}$$
This can be seen as follows. By Hilbert's Theorem 90, elements z such that $N(z) = 1$ are of the form y^q/y for some $y \in \mathbb{F}_{q^2}$, and so
$$\sum_{N(z)=1} \nu(z) = \frac{1}{q} \sum_{y \in \mathbb{F}_{q^2}^\times} \nu(y^q/y) = \frac{1}{q} \sum_{y \in \mathbb{F}_{q^2}^\times} \nu(y)^q \nu(y)^{-1}.$$
Since ν is nondecomposable, there exists $a \in \mathbb{F}_{q^2}^\times$ such that $\nu(a)^q \neq \nu(a)$, i.e., $\nu(a)^q\nu(a)^{-1} \neq 1$. Then,
$$\nu(a)^q\nu(a)^{-1} \sum_{N(z)=1} \nu(z) = \frac{1}{q} \sum_{y \in \mathbb{F}_{q^2}^\times} \nu(ay)^q \nu(ay)^{-1} = \sum_{N(z)=1} \nu(z),$$
and the result follows. Clearly, (2.1) also holds for sums over $N(z) = x$ for any $x \in \mathbb{F}_q^\times$ different from 1 too.

To define the discrete series representation, we use the fact that $\mathrm{GL}_2(\mathbb{F}_q)$ is generated by B and $w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. In fact, when $c \neq 0$, we have the following decomposition
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (bc-ad)/c & -a \\ 0 & -c \end{bmatrix} w \begin{bmatrix} 1 & d/c \\ 0 & 1 \end{bmatrix}.$$

So, we only need to define the representation on B and w. Fix a nontrivial representation ψ of $\mathbb{F}_q$. Then the representation σ_ν acts on the $(q-1)$-dimensional vector space of functions on $\mathbb{F}_q^\times$ as follows:

$$\left(\sigma_\nu \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} f\right)(x) = \nu(d)\psi(bd^{-1}x)f(ad^{-1}x),$$

and

$$\left(\sigma_\nu(w)f\right)(x) = -\sum_{x\in\mathbb{F}_q^\times} \nu(x^{-1})j(yx)f(x),$$

where j is a generalized Kloosterman sum

$$j(u) = \frac{1}{q} \sum_{\substack{t\in\mathbb{F}_q(\sqrt{\delta})^\times \\ N(t)=u}} \psi(t+\bar{t})\nu(t).$$

2.2.1. *Character of σ_ν.*

- Central g:

  $$(\sigma_\nu(g)\delta_a)(x) = \nu(r)\psi(0)\delta_a(x) = \nu(r)\delta_a(x).$$

  So,

  $$\chi_{\sigma_\nu}(g) = (q-1)\nu(r).$$

- Parabolic g:

  $$(\sigma_\nu(g)\delta_a)(x) = \nu(r)\psi(r^{-1}x)\delta_a(x).$$

  Then

  $$\chi_{\sigma_\nu}(g) = \nu(r)\sum_{a\in\mathbb{F}_q^\times}\psi(r^{-1}a) = -\nu(r).$$

- Hyperbolic g:

  $$(\sigma_\nu(g)\delta_a)(x) = \nu(s)\psi(0)\delta_a(rs^{-1}x) = \nu(s)\delta_{sr^{-1}a}(x).$$

  This is a shift matrix not fixing any element, so

  $$\chi_{\sigma_\nu}(g) = 0.$$

- Elliptic g: We use the decomposition into elements of B and w:

  $$\begin{bmatrix} r & s\delta \\ s & r \end{bmatrix} = \begin{bmatrix} (s^2\delta - r^2)/s & -r \\ 0 & -s \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & s^{-1}r \\ 0 & 1 \end{bmatrix}.$$

  We have

  $$\delta_a'(x) := \left(\sigma_\nu \begin{bmatrix} 1 & s^{-1}r \\ 0 & 1 \end{bmatrix}\delta_a\right)(x) = \psi(s^{-1}rx)\delta_a(x),$$

  $$\delta_a''(x) := \left(\sigma_\nu \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\delta_a'\right)(x) = -\sum_{y\in\mathbb{F}_q^\times}\nu(y^{-1})j(xy)\delta_a'(y) = -\nu(a^{-1})j(xa)\psi(s^{-1}ra).$$

  Finally,

  $$\begin{aligned}\left(\sigma_\nu(g)\delta_a\right)(x) &= \left(\sigma_\nu \begin{bmatrix} (s^2\delta - r^2)/s & -r \\ 0 & -s \end{bmatrix}\delta_a''\right)(x) \\ &= \nu(-s)\psi(s^{-1}rx)\delta_a''\left(\frac{r^2 - s^2\delta}{s^2}x\right) \\ &= -\nu\left(-\frac{s}{a}\right)\psi\left(\frac{rx+ra}{s}\right)j\left(\frac{N(z)xa}{s^2}\right),\end{aligned}$$

where we are taking $z = r + s\sqrt{\delta}$ and N is the norm map $\mathbb{F}_{q^2} \to \mathbb{F}_q$. Thus, we calculate the character as

$$\begin{aligned}
\chi_{\sigma_\nu}(g) &= -\sum_{a\in\mathbb{F}_q^\times} \nu\left(-\frac{s}{a}\right)\psi\left(\frac{2ra}{s}\right) j\left(\frac{N(z)a^2}{s^2}\right) \\
&= -\frac{1}{q}\sum_{a\in\mathbb{F}_q^\times} \nu\left(-\frac{s}{a}\right)\psi\left(\frac{2ra}{s}\right) \sum_{\substack{t\in\mathbb{F}_q(\sqrt{\delta})^\times \\ N(t)=a^2N(z)/s^2}} \psi(t+\overline{t})\nu(t) \\
&= -\frac{1}{q}\sum_{a\in\mathbb{F}_q^\times} \psi\left(\frac{2ra}{s}\right) \sum_{\substack{t\in\mathbb{F}_q(\sqrt{\delta})^\times \\ N(t)=a^2N(z)/s^2}} \psi(t+\overline{t})\nu\left(-\frac{s}{a}t\right) \\
&= -\frac{1}{q}\sum_{a\in\mathbb{F}_q^\times} \psi\left(\frac{2ra}{s}\right) \sum_{\substack{t\in\mathbb{F}_q(\sqrt{\delta})^\times \\ N(t)=N(z)}} \psi\left(-\frac{a}{s}(t+\overline{t})\right)\nu(t) \quad \text{(reindexing } t\mapsto -at/s\text{)} \\
&= -\frac{1}{q}\sum_{\substack{t\in\mathbb{F}_q(\sqrt{\delta})^\times \\ N(t)=N(z)}} \nu(t) \sum_{a\in\mathbb{F}_q^\times} \psi\left(\frac{a}{s}(2r-t-\overline{t})\right).
\end{aligned}$$

Now, if $2r-(t+\overline{t}) \neq 0$, then the inner sum is simply -1. Suppose $2r-(t+\overline{t}) = 0$, so then the inner sum is $q-1$; let $t = u+v\sqrt{\delta}$. Then we have $u = r$. But we also require $N(t) = N(z) = r^2 - s^2\delta$, from which we conclude $v = \pm s$, and so $t \in \{z, \overline{z}\}$. Therefore,

$$-q\chi_{\sigma_\nu}(g) = \left(\sum_{\substack{t\in\mathbb{F}_q(\sqrt{\delta})^\times \\ N(t)=N(z)}} -\nu(t)\right) + \nu(z) + \nu(\overline{z}) + (q-1)(\nu(z)+\nu(\overline{z})) = q(\nu(z)+\nu(\overline{z})),$$

where the first sum is zero because ν is nondecomposable (2.1). Thus,

$$\chi_{\sigma_\nu}(g) = -\nu(z) - \nu(\overline{z}).$$